Observability Analysis of Proposed Deterministic Measurement Models for Accelerometers and Magnetometers

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Abstract

This technical report contains mathematical proofs related to the observability of calibration parameters of deterministic measurement error models developed in our related works [1, 2]. We address the observability analysis of the model parameters to support our assumptions and results. To this end, we consider the mathematical conditions required for observability and show that they are satisfied for the measurement error models proposed in our related works.

1 Observability Analysis of the Proposed Accelerometer Model

In [2], we propose an improved measurement model for accelerometers:

$$\vec{a}_m = \underbrace{(\boldsymbol{I}+\boldsymbol{S})\boldsymbol{T}_r^s\boldsymbol{C}_q^r\boldsymbol{C}_p^q\boldsymbol{C}_{\mathrm{NED}}^p}_{\boldsymbol{H}}\,\vec{a}+\vec{b}+\vec{n}. \tag{1}$$

The 13 parameters, which are elements of the parameter vector $\vec{\theta}$ below, are involved in the deterministic part of the measurement model described by Equation (1):

$$\vec{\theta} = \left[\begin{array}{ccccccccccccc} S_x & S_y & S_z & \alpha_1 & \alpha_2 & \alpha_3 & \epsilon_x & \epsilon_y & \epsilon_z & \beta & b_x & b_y & b_z \end{array}\right]^T \tag{2}$$

The vector $\vec{a}_m$ is a recorded accelerometer measurement during the multi-position test. The true excitation signal vector $\vec{a}$ is equal to the gravity vector since the measurements are acquired at stationary positions of the FMS during the multi-position test:

$$\vec{a} = \vec{g}^{\mathrm{NED}} = \left[\begin{array}{ccc} -0.0167 & 0 & 9.7782 \end{array}\right]^T \ \ \mathrm{m/s^2} \tag{3}$$

and $\boldsymbol{C}_p^q$ comprises of ϕ, γ, and ψ angles that determine the trajectory of the FMS during the test. The vector $\vec{a}$ can be parametrized as $\vec{a} = [a_x \ 0 \ a_z]^T$ and $\boldsymbol{C}_p^q$ can be expressed as

$$\boldsymbol{C}_p^q = \begin{bmatrix} \cos\gamma\cos\psi & \cos\gamma\sin\psi & -\sin\gamma \\ \sin\gamma\cos\psi\sin\phi-\sin\psi\cos\phi & \sin\gamma\sin\psi\sin\phi+\cos\psi\cos\phi & \cos\gamma\sin\phi \\ \sin\gamma\cos\psi\cos\phi+\sin\psi\sin\phi & \sin\gamma\sin\psi\cos\phi-\cos\psi\sin\phi & \cos\gamma\cos\phi \end{bmatrix} \tag{4}$$

Note that in the above, $\vec{a}, \vec{a}_m$, and $\boldsymbol{C}_p^q$ are all known. The $\vec{\theta}$ in Equation (2) contains an additional parameter β compared to the traditional accelerometer model parameter vector $\vec{\theta}_{\mathrm{traditional}}$. Hereafter, we investigate whether the parameter set is observable through the proposed model. To this end, we decompose the measurement matrix as $\boldsymbol{H} \triangleq \boldsymbol{G}\boldsymbol{C}_p^q\boldsymbol{C}_{\mathrm{NED}}^p$, where $\boldsymbol{G} \triangleq (\boldsymbol{I}+\boldsymbol{S})\boldsymbol{T}_r^s\boldsymbol{C}_q^r$ is a function of nine parameters, which can be expressed in vectorial form as:

$$\vec{\theta}_- = \left[\begin{array}{ccccccccc} S_x & S_y & S_z & \alpha_1 & \alpha_2 & \alpha_3 & \epsilon_x & \epsilon_y & \epsilon_z \end{array}\right]^T$$

and $\boldsymbol{C}_{\mathrm{NED}}^p$ corresponds to a rotational transformation about the z-axis of the NED frame as

$$\boldsymbol{C}_{\mathrm{NED}}^p = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Since $\boldsymbol{I}+\boldsymbol{S}$ is a diagonal matrix with diagonal elements close to one, $\det(\boldsymbol{T}_r^s) = \cos\alpha_1\cos\alpha_2\sin\alpha_3 \neq 0$ by $\alpha_1 \approx 0$, $\alpha_2 \approx 0$, and $\alpha_3 \approx \pi/2$, and $\left(\boldsymbol{C}_q^r\right)^{-1} = \left(\boldsymbol{C}_q^r\right)^T$, the matrix $\boldsymbol{G}$ is invertible. Thus, $\boldsymbol{G}$ is the image of a full rank mapping as $\vec{\theta}_- \in \mathbb{R}^9 \to \boldsymbol{G} \in \mathbb{R}^3 \times \mathbb{R}^3$ and we can treat the elements of $\boldsymbol{G}$ as independent parameters g_{ij} instead of providing their explicit relations with the elements of $\vec{\theta}_{\mathrm{traditional}}$ for the observability analysis. Having defined $\boldsymbol{G}$, $\boldsymbol{C}_p^q$, and $\boldsymbol{C}_{\mathrm{NED}}^p$, we rewrite the measurement equation [Equation (1)] more explicitly in component-wise form as

$$\vec{a}_m = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} \cos\gamma\cos\psi & \cos\gamma\sin\psi & -\sin\gamma \\ \sin\gamma\cos\psi\sin\phi-\sin\psi\cos\phi & \sin\gamma\sin\psi\sin\phi+\cos\psi\cos\phi & \cos\gamma\sin\phi \\ \sin\gamma\cos\psi\cos\phi+\sin\psi\sin\phi & \sin\gamma\sin\psi\cos\phi-\cos\psi\sin\phi & \cos\gamma\cos\phi \end{bmatrix}$$

$$
\times \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ 0 \\ a_z \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \tag{6}
$$

We now rearrange Equation (6) in a way that parameters that belong to $\vec{\theta}$ are collected in a single matrix of unknowns, while known variables, which are a_x, a_z, ϕ, γ, and ψ, are collected under a single vector. Then, we will show that the matrix of unknowns is a function of 13 parameters in $\vec{\theta}$. To this end, we first multiply $\boldsymbol{C}_p^q$ and $\boldsymbol{C}_{\mathrm{NED}}^p$ with $\vec{a}$ and rewrite Equation (6) as follows:

$$
\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \left(a_x \begin{bmatrix} \cos\beta\cos\gamma\cos\psi - \sin\beta\cos\gamma\sin\psi \\ \cos\beta\left(\sin\gamma\cos\psi\sin\phi - \sin\psi\cos\phi\right) + \sin\beta\left(\cos\psi\cos\phi + \sin\gamma\sin\psi\sin\phi\right) \\ \cos\beta\left(\sin\gamma\cos\psi\cos\phi + \sin\psi\sin\phi\right) + \sin\beta\left(\cos\psi\sin\phi - \sin\gamma\sin\psi\cos\phi\right) \end{bmatrix} \right.
$$

$$
\left. + a_z \begin{bmatrix} -\sin\gamma \\ \cos\gamma\sin\phi \\ \cos\gamma\cos\phi \end{bmatrix} \right) + \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \vec{a}_m. \tag{7}
$$

Further rearrangements result in the following:

$$
\overbrace{\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \left[\begin{array}{ccc|ccc|ccc} \cos\beta & 0 & 0 & \sin\beta & 0 & 0 & 1 & 0 & 0 \\ 0 & \cos\beta & 0 & 0 & \sin\beta & 0 & 0 & 1 & 0 \\ 0 & 0 & \cos\beta & 0 & 0 & \sin\beta & 0 & 0 & 1 \end{array}\right]}^{\boldsymbol{H}'} \overbrace{\left[\begin{array}{c} a_x\cos\gamma\cos\psi \\ a_x\left(\sin\gamma\cos\psi\sin\phi - \sin\psi\cos\phi\right) \\ a_x\left(\sin\gamma\cos\psi\cos\phi + \sin\psi\sin\phi\right) \\ \hline -a_x\cos\gamma\sin\psi \\ a_x\left(\cos\psi\cos\phi + \sin\gamma\sin\psi\sin\phi\right) \\ a_x\left(\cos\psi\sin\phi - \sin\gamma\sin\psi\cos\phi\right) \\ \hline -a_z\sin\gamma \\ a_z\cos\gamma\sin\phi \\ a_z\cos\gamma\cos\phi \end{array}\right]}^{\vec{a}'}
$$

$$
+ \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \vec{a}_m. \tag{8}
$$

The extended measurement matrix $\boldsymbol{H}'$ in Equation (8) is given by:

$$
\boldsymbol{H}' = \left[\begin{array}{ccc|ccc|ccc} g_{11}\cos\beta & g_{12}\cos\beta & g_{13}\cos\beta & g_{11}\sin\beta & g_{12}\sin\beta & g_{13}\sin\beta & g_{11} & g_{12} & g_{13} \\ g_{21}\cos\beta & g_{22}\cos\beta & g_{23}\cos\beta & g_{21}\sin\beta & g_{22}\sin\beta & g_{23}\sin\beta & g_{21} & g_{22} & g_{23} \\ g_{31}\cos\beta & g_{32}\cos\beta & g_{33}\cos\beta & g_{31}\sin\beta & g_{32}\sin\beta & g_{33}\sin\beta & g_{31} & g_{32} & g_{33} \end{array}\right], \tag{9}
$$

where $\boldsymbol{H}'$ depends on all the parameters in $\vec{\theta}$ except for b_x, b_y, and b_z.

In total, we have 13 independent parameters to be estimated (nine parameters in $\boldsymbol{G}$, three in $\vec{b}$, and β) without any further dependencies between them. According to the rearranged model, 10 of these parameters are involved in $\boldsymbol{H}'$ and three of them come from $\vec{b}$. In our work, we perform the calibration based on N measurements. Every measurement results in three equations, and after acquiring N measurements, we obtain $3N$ equations to solve for the unknown parameter vector with 13 elements. However, the equations are nonlinear.

In order to check the degrees of freedom (DoF) of our estimation/calibration problem, it is not sufficient to investigate the number of underlying independent parameters forming $\boldsymbol{H}'$ but the whole set of measurement and excitation signals throughout the experiment. In terms of measurement and excitation signals, we ensure that our calibration procedure is sufficiently long and comprehensive that we do not loose any DoF of the original measurement model in our measurements. Underdetermined systems where there are fewer measurements than unknowns can be considered to be a troublesome case.

After taking the transpose of Equation (8), concatenating the rows of $\boldsymbol{H}'$ to form a 27×1 vector and augmenting that vector by the bias vector elements into a 30×1 vector $\vec{\theta}^e$, we can represent our N measurements using a single vector-matrix equation in the form of $\vec{y} = \boldsymbol{A}\vec{\theta}^e$:

$$\underbrace{\begin{bmatrix} \vec{a}_m[1] \\ \vec{a}_m[2] \\ \vdots \\ \vec{a}_m[N] \end{bmatrix}}_{\vec{y}} = \underbrace{\left[\begin{array}{c:c:c:ccc} \vec{a'}^T[1] & \vec{0}^T & \vec{0}^T & 1 & 0 & 0 \\ \vec{0}^T & \vec{a'}^T[1] & \vec{0}^T & 0 & 1 & 0 \\ \vec{0}^T & \vec{0}^T & \vec{a'}^T[1] & 0 & 0 & 1 \\ \hdashline \vec{a'}^T[2] & \vec{0}^T & \vec{0}^T & 1 & 0 & 0 \\ \vec{0}^T & \vec{a'}^T[2] & \vec{0}^T & 0 & 1 & 0 \\ \vec{0}^T & \vec{0}^T & \vec{a'}^T[2] & 0 & 0 & 1 \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline \vec{a'}^T[N] & \vec{0}^T & \vec{0}^T & 1 & 0 & 0 \\ \vec{0}^T & \vec{a'}^T[N] & \vec{0}^T & 0 & 1 & 0 \\ \vec{0}^T & \vec{0}^T & \vec{a'}^T[N] & 0 & 0 & 1 \end{array}\right]}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} g_{11}\cos\beta \\ g_{12}\cos\beta \\ g_{13}\cos\beta \\ g_{11}\sin\beta \\ g_{12}\sin\beta \\ g_{13}\sin\beta \\ g_{11} \\ g_{12} \\ g_{13} \\ g_{21}\cos\beta \\ g_{22}\cos\beta \\ g_{23}\cos\beta \\ g_{21}\sin\beta \\ g_{22}\sin\beta \\ g_{23}\sin\beta \\ g_{21} \\ g_{22} \\ g_{23} \\ g_{31}\cos\beta \\ g_{32}\cos\beta \\ g_{33}\cos\beta \\ g_{31}\sin\beta \\ g_{32}\sin\beta \\ g_{33}\sin\beta \\ g_{31} \\ g_{32} \\ g_{33} \\ b_x \\ b_y \\ b_z \end{bmatrix}}_{\vec{\theta}^e} \tag{10}$$

Here, $\vec{0}$ is the 9×1 vector of zeros. The least-squares solution $\arg\min_{\vec{\theta}} \|\vec{y} - \boldsymbol{A}\vec{\theta}^e\|$ to this problem is equivalent to the minimization problem $\arg\min_{\vec{\theta}} \|\vec{y} - \vec{G}(\vec{\theta})\|$ in our article. In order to show that all of the 13 parameters are observable, we need to show two things: 1) The rank of $\boldsymbol{A}$ needs to be at least 13 [3]; 2) $\vec{\theta}^e$ needs to have a dimension of at least 13 [4]. Since $N \gg 30$ and ϕ, γ, ψ take many different values (more than 13) covering the range $[-\pi, +\pi)$ during the calibration procedure, $\text{rank}(\boldsymbol{A}) > 13$. As for the second condition, we need to check the rank of the mapping $M: \vec{\theta} \in \mathbb{R}^{13} \rightarrow \vec{\theta}^e \in \mathbb{R}^{30}$, since this rank determines the actual dimension of $\vec{\theta}^e$. If the rank of the mapping M is equal to 13, the underlying DoFs in the solution space is 13 as well, meaning that there exists a unique $\vec{\theta}$ solution. (A similar observability analysis is performed in [5].) For this analysis, assuming that $\boldsymbol{G}$ has nine DoFs, we take the modified $\vec{\theta}$ as $\vec{\theta} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{21} & g_{22} & g_{23} & g_{31} & g_{32} & g_{33} & \beta & b_x & b_y & b_z \end{bmatrix}^T$.

The mapping M between $\vec{\theta}$ and $\vec{\theta}^e$ can be described by 30 multi-variable nonlinear functions such that

$$
\begin{aligned}
\theta_1^e &\equiv f_1(\theta_1, \theta_2, \ldots, \theta_{13}) = \theta_1 \cos\theta_{10} \\
\theta_2^e &\equiv f_2(\theta_1, \theta_2, \ldots, \theta_{13}) = \theta_2 \cos\theta_{10} \\
\theta_3^e &\equiv f_3(\theta_1, \theta_2, \ldots, \theta_{13}) = \theta_3 \cos\theta_{10} \\
&\vdots \\
\theta_{30}^e &\equiv f_{30}(\theta_1, \theta_2, \ldots, \theta_{13}) = \theta_{13},
\end{aligned}
\tag{11}
$$

where θ_i^e, θ_i, and f_is correspond to the ith component of $\vec{\theta}^e$, modified $\vec{\theta}$, and the nonlinear functions representing M, respectively.

The rank of M can be determined by deriving its 30×13 Jacobian matrix through the following Jacobian definition based on the notation of Equation (11):

$$
\boldsymbol{J}_M = \begin{bmatrix}
\frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} & \frac{\partial f_1}{\partial \theta_3} & \cdots & \frac{\partial f_1}{\partial \theta_{13}} \\
\frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} & \frac{\partial f_2}{\partial \theta_3} & \cdots & \frac{\partial f_2}{\partial \theta_{13}} \\
\frac{\partial f_3}{\partial \theta_1} & \frac{\partial f_3}{\partial \theta_2} & \frac{\partial f_3}{\partial \theta_3} & \cdots & \frac{\partial f_3}{\partial \theta_{13}} \\
 & & & \vdots & \\
\frac{\partial f_{30}}{\partial \theta_1} & \frac{\partial f_{30}}{\partial \theta_2} & \frac{\partial f_{30}}{\partial \theta_3} & \cdots & \frac{\partial f_{30}}{\partial \theta_{13}}
\end{bmatrix}
\tag{12}
$$

$$
= \begin{bmatrix}
\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_{11}\sin\beta & 0 & 0 & 0 \\
0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_{12}\sin\beta & 0 & 0 & 0 \\
0 & 0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & -g_{13}\sin\beta & 0 & 0 & 0 \\
\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{11}\cos\beta & 0 & 0 & 0 \\
0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{12}\cos\beta & 0 & 0 & 0 \\
0 & 0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & g_{13}\cos\beta & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & -g_{21}\sin\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos\beta & 0 & 0 & 0 & 0 & -g_{22}\sin\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cos\beta & 0 & 0 & 0 & -g_{23}\sin\beta & 0 & 0 & 0 \\
0 & 0 & 0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & g_{21}\cos\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sin\beta & 0 & 0 & 0 & 0 & g_{22}\cos\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sin\beta & 0 & 0 & 0 & g_{23}\cos\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cos\beta & 0 & 0 & -g_{31}\sin\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\beta & 0 & -g_{32}\sin\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\beta & -g_{33}\sin\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & 0 & 0 & g_{31}\cos\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & 0 & g_{32}\cos\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & g_{33}\cos\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Inspection of $\boldsymbol{J}_M$ reveals that its rank is 13 since $\cos\beta$ and $\sin\beta$ cannot be zero at the same time. With this interpretation, we complete the above steps and conclude that our minimization problem $\arg\min_{\vec{\theta}} \|\vec{y} - \vec{G}(\vec{\theta})\|$ has 13 DoFs.

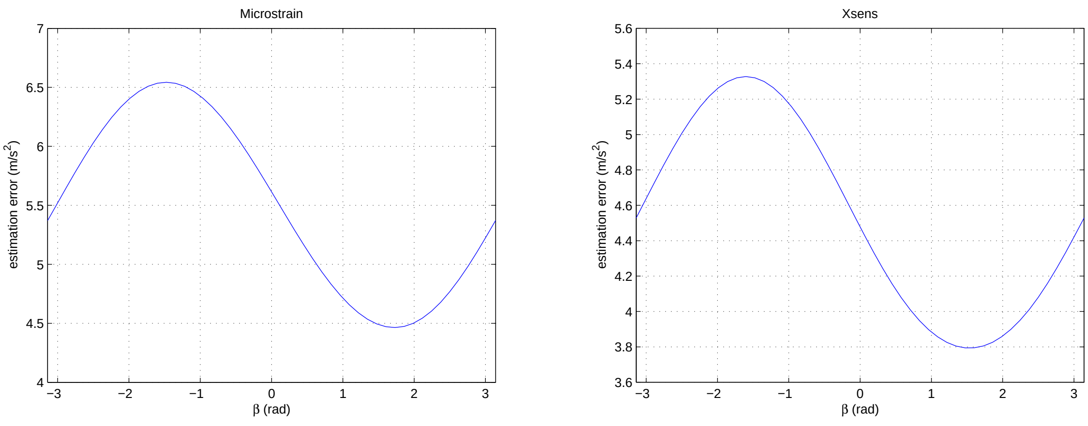
Another and probably a less tedious way to show that our calibration problem has 13 DoFs is to incorporate different sample-based $\boldsymbol{C}_p^q$ matrices into our proposed model; thus, increasing the dimensionality of our measurement equation in a different way. To this end, we write the following N-sample measurement equation:

$$\underbrace{\begin{bmatrix} \vec{a}_m[1] \\ \vec{a}_m[2] \\ \vdots \\ \vec{a}_m[N] \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} \boldsymbol{G} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{G} & \mathbf{0} & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}_p^q[1] \\ \boldsymbol{C}_p^q[2] \\ \vdots \\ \boldsymbol{C}_p^q[N] \end{bmatrix} \boldsymbol{C}_{\mathrm{NED}}^p \vec{a}}_{\boldsymbol{H}''} + \begin{bmatrix} \vec{b} \\ \vec{b} \\ \vdots \\ \vec{b} \end{bmatrix} \tag{13}$$

Since the dimension of $\boldsymbol{H}''$ is $N \times 3$ in this augmented form, the maximum possible DoFs of $\boldsymbol{H}''$ is also $3N$. In order to see the exact number of DoFs of this augmented form, one needs to carry out an analysis as in the first approach. Based on this, one can claim that we do not need to limit the dimension of $\vec{\theta}$ to 12 and can use a parameter vector $\vec{\theta}$ of length 13.

In both of the above approaches, we express the measurement model in a higher-dimensional space, indicating that each of the parameters in $\vec{\theta}$ contributes to a unique measurement matrix and none of them are redundant.

We also evaluate the validity of our conclusion using the acquired accelerometer data as follows: Given a value for $\beta \in [-\pi, \pi)$, we perform the calibration based on other parameters and investigate the relationship between β and achieved estimation error. By this, we can decide whether β is a redundant parameter or our parameter vector $\vec{\theta}$ with 13 elements is overdetermined. Results of this investigation are provided in Figure 1. As the figure illustrates, the estimation performance heavily depends on the value of β, thus proving us that β is a not a redundant parameter and needs to be contained in the parameter vector $\vec{\theta}$ in order to capture the proposed model completely.



(a) (b)

Figure 1: Dependence of estimation accuracy with respect to β for (a) MicroStrain and (b) Xsens accelerometers.

2 Observability Analysis of the Traditional Magnetometer Model

In the original manuscript, we presented the traditional magnetometer model as

$$\vec{B}_m = (\boldsymbol{I} + \boldsymbol{S})\, \boldsymbol{T}_r^s \left(\boldsymbol{K} \boldsymbol{C}_q^r \boldsymbol{C}_p^q \boldsymbol{C}_{\text{NED}}^p \, \vec{B}^{\text{NED}} + \delta\vec{B} \right) + \vec{b} + \vec{n} \tag{14}$$

and estimate the parameter vector $\vec{\theta}$ corresponding to the equation given above. We begin with correcting the traditional magnetometer model given above as

$$\vec{B}_m = (\boldsymbol{I} + \boldsymbol{S})\, \boldsymbol{T}_r^s \boldsymbol{C}_q^r \left(\boldsymbol{K} \boldsymbol{C}_p^q \boldsymbol{C}_{\text{NED}}^p \, \vec{B}^{\text{NED}} + \delta\vec{B} \right) + \vec{b} + \vec{n}. \tag{15}$$

If we perform a similar decomposition as in the accelerometer case with $\boldsymbol{H} \triangleq \boldsymbol{G} \boldsymbol{C}_p^q \boldsymbol{C}_{\text{NED}}^p$ and $\boldsymbol{G} \triangleq (\boldsymbol{I} + \boldsymbol{S}) \boldsymbol{T}_r^s \boldsymbol{C}_q^r \boldsymbol{K}$, we can observe that unlike the accelerometer case, $\boldsymbol{G}$ is a function of 15 parameters (three parameters in $\boldsymbol{S}, \boldsymbol{T}_r^s$, and $\boldsymbol{C}_q^r$ each, and six in $\boldsymbol{K}$). However, given that the $\boldsymbol{G}$ matrix contains nine elements, only nine of these 15 parameters can be independent. It is not possible to observe and estimate all of the underlying 15 parameters. This can be further clarified by considering a mapping

$$M : \vec{\theta} \in \mathbb{R}^{15} \rightarrow \left[\begin{array}{ccccccccc} g_{11} & g_{12} & g_{13} & g_{21} & g_{22} & g_{23} & g_{31} & g_{32} & g_{33} \end{array} \right] \in \mathbb{R}^9.$$

Since the Jacobian $\boldsymbol{J}_M$ of M is a 9×15 matrix, the rank of $\boldsymbol{J}_M$ for this case cannot be larger than nine and only nine of the 15 parameters can be estimated.

To remedy the problem, we rewrite the measurement model according to the unified measurement model given by Equation (10) in our article as:

$$\vec{B}_m = \boldsymbol{G} \boldsymbol{C}_p^q \boldsymbol{C}_{\text{NED}}^p \vec{B}^{\text{NED}} + \vec{b}' + \vec{n}. \tag{16}$$

where $\vec{b}' = (\boldsymbol{I} + \boldsymbol{S})\, \boldsymbol{T}_r^s \, \boldsymbol{C}_q^r \, \delta\vec{B} + \vec{b}$. Equation (16) is in exactly the same form as the accelerometer measurement equation. We also modify the parameter vector so that it contains 13 elements:

$$\vec{\theta} = \left[\begin{array}{ccccccccccccc} g_{11} & g_{12} & g_{13} & g_{21} & g_{22} & g_{23} & g_{31} & g_{32} & g_{33} & \beta & b'_x & b'_y & b'_z \end{array} \right]^T \tag{17}$$

Following the same steps as in the accelerometer case, we can show that the 13 parameters that comprise the modified parameter vector are observable. We have modified the original manuscript accordingly and provided updated parameter estimation results in Tables 8 and 9 of the manuscript for magnetometers.

3 Observability Analysis of the Proposed Magnetometer Model

In the original manuscript, the proposed magnetometer model was given as:

$$\vec{B}_m = (\boldsymbol{I} + \boldsymbol{S})\, \boldsymbol{T}_r^s \left(\boldsymbol{K} \boldsymbol{C}_q^r \boldsymbol{C}_p^q \boldsymbol{C}_{\text{NED}}^p \vec{B}^{\text{NED}} + \boldsymbol{C}_q^r \boldsymbol{C}_p^q \, \delta\vec{B} \right) + \vec{b} + \vec{n}. \tag{18}$$

The change in the traditional magnetometer model leads to a natural modification to the proposed magnetometer model as

$$\vec{B}_m = (\boldsymbol{I} + \boldsymbol{S})\, \boldsymbol{T}_r^s \boldsymbol{C}_q^r \left(\boldsymbol{K} \boldsymbol{C}_p^q \boldsymbol{C}_{\text{NED}}^p \, \vec{B}^{\text{NED}} + \boldsymbol{C}_p^q \delta\vec{B} \right) + \vec{b} + \vec{n}. \tag{19}$$

Defining $\boldsymbol{V} \triangleq (\boldsymbol{I}+\boldsymbol{S})\boldsymbol{T}_r^s\boldsymbol{C}_q^r$, we can rewrite the above equation as

$$\vec{B}_m = \boldsymbol{V}\boldsymbol{K}\boldsymbol{C}_p^q\boldsymbol{C}_{\mathrm{NED}}^p\,\vec{B}^{\mathrm{NED}} + \boldsymbol{V}\boldsymbol{C}_p^q\delta\vec{B} + \vec{b} + \vec{n}. \tag{20}$$

We express $\boldsymbol{G}$ as $\boldsymbol{G} = \boldsymbol{V}\boldsymbol{K}$ as in Section 2 and after noting that $\boldsymbol{K}$ is symmetric, $\boldsymbol{V} = \boldsymbol{G}\underline{\boldsymbol{K}}$ where $\underline{\boldsymbol{K}} = \boldsymbol{K}^{-1}$ is also a symmetric matrix since $\boldsymbol{K}^{-1} = \left(\boldsymbol{K}^T\right)^{-1} = \left(\boldsymbol{K}^{-1}\right)^T$. We reformulate Equation (20) based on these definitions as

$$\vec{B}_m = \boldsymbol{G}\boldsymbol{C}_p^q\boldsymbol{C}_{\mathrm{NED}}^p\,\vec{B}^{\mathrm{NED}} + \boldsymbol{G}\underline{\boldsymbol{K}}\boldsymbol{C}_p^q\delta\vec{B} + \vec{b} + \vec{n}. \tag{21}$$

As explained in Section 2, we represent the $\boldsymbol{G}$ matrix by nine parameters corresponding to each of its elements as

$$\boldsymbol{G} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}.$$

As for $\underline{\boldsymbol{K}}$, we only need six parameters since it is a symmetric matrix as follows:

$$\underline{\boldsymbol{K}} = \begin{bmatrix} \underline{k}_{11} & \underline{k}_{12} & \underline{k}_{13} \\ \underline{k}_{12} & \underline{k}_{22} & \underline{k}_{23} \\ \underline{k}_{13} & \underline{k}_{23} & \underline{k}_{33} \end{bmatrix}.$$

The Earth's magnetic field at the location of the experiments is given by:

$$\vec{B}^{\mathrm{NED}} = \begin{bmatrix} 0.2523 & 0.0217 & 0.4004 \end{bmatrix}^T \text{ Gauss}, \tag{22}$$

which can be parametrized by $\vec{B}^{\mathrm{NED}} = [m_x\ m_y\ m_z]^T$. With these definitions, we put the proposed measurement equation Equation (21) into the component-wise form as

$$\vec{B}_m = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} \cos\gamma\cos\psi & \cos\gamma\sin\psi & -\sin\gamma \\ \sin\gamma\cos\psi\sin\phi - \sin\psi\cos\phi & \sin\gamma\sin\psi\sin\phi + \cos\psi\cos\phi & \cos\gamma\sin\phi \\ \sin\gamma\cos\psi\cos\phi + \sin\psi\sin\phi & \sin\gamma\sin\psi\cos\phi - \cos\psi\sin\phi & \cos\gamma\cos\phi \end{bmatrix}$$
$$\times \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} \underline{k}_{11} & \underline{k}_{12} & \underline{k}_{13} \\ \underline{k}_{12} & \underline{k}_{22} & \underline{k}_{23} \\ \underline{k}_{13} & \underline{k}_{23} & \underline{k}_{33} \end{bmatrix}$$
$$\times \begin{bmatrix} \cos\gamma\cos\psi & \cos\gamma\sin\psi & -\sin\gamma \\ \sin\gamma\cos\psi\sin\phi - \sin\psi\cos\phi & \sin\gamma\sin\psi\sin\phi + \cos\psi\cos\phi & \cos\gamma\sin\phi \\ \sin\gamma\cos\psi\cos\phi + \sin\psi\sin\phi & \sin\gamma\sin\psi\cos\phi - \cos\psi\sin\phi & \cos\gamma\cos\phi \end{bmatrix} \begin{bmatrix} \delta B_x \\ \delta B_y \\ \delta B_z \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \tag{23}$$

Similar to the observability analysis of accelerometers, we separate the known and unknown variables and collect them in a single vector and matrix, respectively. We follow the same steps as in Section 1.

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \left(m_x \begin{bmatrix} \cos\beta\cos\gamma\cos\psi - \sin\beta\cos\gamma\sin\psi \\ \cos\beta\left(\sin\gamma\cos\psi\sin\phi - \sin\psi\cos\phi\right) + \sin\beta\left(\cos\psi\cos\phi + \sin\gamma\sin\psi\sin\phi\right) \\ \cos\beta\left(\sin\gamma\cos\psi\cos\phi + \sin\psi\sin\phi\right) + \sin\beta\left(\cos\psi\sin\phi - \sin\gamma\sin\psi\cos\phi\right) \end{bmatrix} \right.$$
$$+ m_y \begin{bmatrix} \cos\beta\cos\gamma\sin\psi + \sin\beta\cos\gamma\cos\psi \\ \cos\beta\left(\sin\gamma\sin\psi\sin\phi + \cos\psi\cos\phi\right) + \sin\beta\left(\sin\gamma\cos\psi\sin\phi - \sin\psi\cos\phi\right) \\ \cos\beta\left(\sin\gamma\sin\psi\cos\phi - \cos\psi\sin\phi\right) + \sin\beta\left(\sin\gamma\cos\psi\cos\phi + \sin\psi\sin\phi\right) \end{bmatrix}$$

$$+m_z \begin{bmatrix} -\sin\gamma \\ \cos\gamma\sin\phi \\ \cos\gamma\cos\phi \end{bmatrix} \Bigg)$$

$$+ \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} \underline{k}_{11} & \underline{k}_{12} & \underline{k}_{13} \\ \underline{k}_{12} & \underline{k}_{22} & \underline{k}_{23} \\ \underline{k}_{13} & \underline{k}_{23} & \underline{k}_{33} \end{bmatrix} \Bigg(\delta B_x \begin{bmatrix} \cos\gamma\cos\psi \\ \sin\gamma\cos\psi\sin\phi - \sin\psi\cos\phi \\ \sin\gamma\cos\psi\cos\phi + \sin\psi\sin\phi \end{bmatrix}$$

$$+\delta B_y \begin{bmatrix} \cos\gamma\sin\psi \\ \sin\gamma\sin\psi\sin\phi + \cos\psi\cos\phi \\ \sin\gamma\sin\psi\cos\phi - \cos\psi\sin\phi \end{bmatrix}$$

$$+\delta B_z \begin{bmatrix} -\sin\gamma \\ \cos\gamma\sin\phi \\ \cos\gamma\cos\phi \end{bmatrix} \Bigg) = \vec{B}_m \tag{24}$$

Further manipulation yields the following:

$$\vec{B}_m = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \left[\begin{array}{ccc:ccc:ccc} \cos\beta & 0 & 0 & \sin\beta & 0 & 0 & 1 & 0 & 0 \\ 0 & \cos\beta & 0 & 0 & \sin\beta & 0 & 0 & 1 & 0 \\ 0 & 0 & \cos\beta & 0 & 0 & \sin\beta & 0 & 0 & 1 \end{array}\right]$$

$$\times \underbrace{\left[\begin{array}{c} m_x\cos\gamma\cos\psi + m_y\cos\gamma\sin\psi \\ m_x\left(\sin\gamma\cos\psi\sin\phi - \sin\psi\cos\phi\right) + m_y\left(\sin\gamma\sin\psi\sin\phi + \cos\psi\cos\phi\right) \\ m_x\left(\sin\gamma\cos\psi\cos\phi + \sin\psi\sin\phi\right) + m_y\left(\sin\gamma\sin\psi\cos\phi - \cos\psi\sin\phi\right) \\ \hdashline -m_x\cos\gamma\sin\psi + m_y\cos\gamma\cos\psi \\ m_x\left(\cos\psi\cos\phi + \sin\gamma\sin\psi\sin\phi\right) + m_y\left(\sin\gamma\cos\psi\sin\phi - \sin\psi\cos\phi\right) \\ m_x\left(\cos\psi\sin\phi - \sin\gamma\sin\psi\cos\phi\right) + m_y\left(\sin\gamma\cos\psi\cos\phi + \sin\psi\sin\phi\right) \\ \hdashline -m_z\sin\gamma \\ m_z\cos\gamma\sin\phi \\ m_z\cos\gamma\cos\phi \end{array}\right]}_{\vec{m}'_1}$$

$$+ \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} \underline{k}_{11} & \underline{k}_{12} & \underline{k}_{13} \\ \underline{k}_{12} & \underline{k}_{22} & \underline{k}_{23} \\ \underline{k}_{13} & \underline{k}_{23} & \underline{k}_{33} \end{bmatrix} \left[\begin{array}{ccc:ccc:ccc} \delta B_x & 0 & 0 & \delta B_y & 0 & 0 & \delta B_z & 0 & 0 \\ 0 & \delta B_x & 0 & 0 & \delta B_y & 0 & 0 & \delta B_z & 0 \\ 0 & 0 & \delta B_x & 0 & 0 & \delta B_x & 0 & 0 & \delta B_z \end{array}\right]$$

$$\times \underbrace{\left[\begin{array}{c} \cos\gamma\cos\psi \\ \sin\gamma\cos\psi\sin\phi - \sin\psi\cos\phi \\ \sin\gamma\cos\psi\cos\phi + \sin\psi\sin\phi \\ \hdashline \cos\gamma\sin\psi \\ (\cos\psi\cos\phi + \sin\gamma\sin\psi\sin\phi \\ \sin\gamma\sin\psi\cos\phi - \cos\psi\sin\phi \\ \hdashline -\sin\gamma \\ \cos\gamma\sin\phi \\ \cos\gamma\cos\phi \end{array}\right]}_{\vec{m}'_2} = \vec{B}_m. \tag{25}$$

Now, we define another two matrices as

$$\boldsymbol{H}'_1 = \boldsymbol{G} \left[\begin{array}{ccc:ccc:ccc} \cos\beta & 0 & 0 & \sin\beta & 0 & 0 & 1 & 0 & 0 \\ 0 & \cos\beta & 0 & 0 & \sin\beta & 0 & 0 & 1 & 0 \\ 0 & 0 & \cos\beta & 0 & 0 & \sin\beta & 0 & 0 & 1 \end{array}\right]$$

$$\boldsymbol{H}_2' = \boldsymbol{GK}\left[\begin{array}{ccc:ccc:ccc} \delta B_x & 0 & 0 & \delta B_y & 0 & 0 & \delta B_z & 0 & 0 \\ 0 & \delta B_x & 0 & 0 & \delta B_y & 0 & 0 & \delta B_z & 0 \\ 0 & 0 & \delta B_x & 0 & 0 & \delta B_x & 0 & 0 & \delta B_z \end{array}\right].$$

To be clear, we remind that $\boldsymbol{H}_1'$ and $\boldsymbol{H}_2'$ are unknown calibration matrices while $\vec{m}_1'$ and $\vec{m}_2'$ are known signals. The expanded forms of $\boldsymbol{H}_1'$ and $\boldsymbol{H}_2'$ can be written as:

$$\boldsymbol{H}_1' = \left[\begin{array}{ccc:ccc:ccc} g_{11}\cos\beta & g_{12}\cos\beta & g_{13}\cos\beta & g_{11}\sin\beta & g_{12}\sin\beta & g_{13}\sin\beta & g_{11} & g_{12} & g_{13} \\ g_{21}\cos\beta & g_{22}\cos\beta & g_{23}\cos\beta & g_{21}\sin\beta & g_{22}\sin\beta & g_{23}\sin\beta & g_{21} & g_{22} & g_{23} \\ g_{31}\cos\beta & g_{32}\cos\beta & g_{33}\cos\beta & g_{31}\sin\beta & g_{32}\sin\beta & g_{33}\sin\beta & g_{31} & g_{32} & g_{33} \end{array}\right] \tag{26}$$

and

$$\boldsymbol{H}_2' = \left[\begin{array}{ccc:ccc:ccc} \delta B_x w_{11} & \delta B_x w_{12} & \delta B_x w_{13} & \delta B_y w_{11} & \delta B_y w_{12} & \delta B_y w_{13} & \delta B_z w_{11} & \delta B_z w_{12} & \delta B_z w_{13} \\ \delta B_x w_{21} & \delta B_x w_{22} & \delta B_x w_{23} & \delta B_y w_{21} & \delta B_y w_{22} & \delta B_y w_{23} & \delta B_z w_{21} & \delta B_z w_{22} & \delta B_z w_{23} \\ \delta B_x w_{31} & \delta B_x w_{32} & \delta B_x w_{33} & \delta B_y w_{31} & \delta B_y w_{32} & \delta B_y w_{33} & \delta B_z w_{31} & \delta B_z w_{32} & \delta B_z w_{33} \end{array}\right] \tag{27}$$

where

$$\begin{aligned} w_{11} &= g_{11}\underline{k}_{11} + g_{12}\underline{k}_{12} + g_{13}\underline{k}_{13} \\ w_{12} &= g_{11}\underline{k}_{12} + g_{12}\underline{k}_{22} + g_{13}\underline{k}_{23} \\ w_{13} &= g_{11}\underline{k}_{13} + g_{12}\underline{k}_{23} + g_{13}\underline{k}_{33} \\ w_{21} &= g_{21}\underline{k}_{11} + g_{22}\underline{k}_{12} + g_{23}\underline{k}_{13} \\ w_{22} &= g_{21}\underline{k}_{12} + g_{22}\underline{k}_{22} + g_{23}\underline{k}_{23} \\ w_{23} &= g_{21}\underline{k}_{13} + g_{22}\underline{k}_{23} + g_{23}\underline{k}_{33} \\ w_{31} &= g_{31}\underline{k}_{11} + g_{32}\underline{k}_{12} + g_{33}\underline{k}_{13} \\ w_{32} &= g_{31}\underline{k}_{12} + g_{32}\underline{k}_{22} + g_{33}\underline{k}_{23} \\ w_{33} &= g_{31}\underline{k}_{13} + g_{32}\underline{k}_{23} + g_{33}\underline{k}_{33}. \end{aligned}$$

After taking the transpose of $\boldsymbol{H}_1'$ and $\boldsymbol{H}_2'$, concatenating their rows to form 27×1 vectors and augmenting those vectors with the bias vector elements into a 57×1 vector $\vec{\theta}^e$, we can represent our N measurements using a single vector-matrix equation in the form $\vec{y} = \boldsymbol{A}\vec{\theta}^e$:

$$\underbrace{\begin{bmatrix}\vec{B}_m[1]\\ \vec{B}_m[2]\\ \vdots\\ \vec{B}_m[N]\end{bmatrix}}_{\vec{y}} = \underbrace{\left[\begin{array}{c|c|c|c|c|c|ccc}
\vec{m'}^T_1[1] & \vec{0}^T & \vec{0}^T & \vec{m'}^T_2[1] & \vec{0}^T & \vec{0}^T & 1 & 0 & 0\\
\vec{0}^T & \vec{m'}^T_1[1] & \vec{0}^T & \vec{0}^T & \vec{m'}^T_2[1] & \vec{0}^T & 0 & 1 & 0\\
\vec{0}^T & \vec{0}^T & \vec{m'}^T_1[1] & \vec{0}^T & \vec{0}^T & \vec{m'}^T_2[1] & 0 & 0 & 1\\
\hline
\vec{m'}^T_1[2] & \vec{0}^T & \vec{0}^T & \vec{m'}^T_2[2] & \vec{0}^T & \vec{0}^T & 1 & 0 & 0\\
\vec{0}^T & \vec{m'}^T_1[2] & \vec{0}^T & \vec{0}^T & \vec{m'}^T_2[2] & \vec{0}^T & 0 & 1 & 0\\
\vec{0}^T & \vec{0}^T & \vec{m'}^T_1[2] & \vec{0}^T & \vec{0}^T & \vec{m'}^T_2[2] & 0 & 0 & 1\\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\hline
\vec{m'}^T_1[N] & \vec{0}^T & \vec{0}^T & \vec{m'}^T_2[N] & \vec{0}^T & \vec{0}^T & 1 & 0 & 0\\
\vec{0}^T & \vec{m'}^T_1[N] & \vec{0}^T & \vec{0}^T & \vec{m'}^T_2[N] & \vec{0}^T & 0 & 1 & 0\\
\vec{0}^T & \vec{0}^T & \vec{m'}^T_1[N] & \vec{0}^T & \vec{0}^T & \vec{m'}^T_2[N] & 0 & 0 & 1
\end{array}\right]}_{\boldsymbol{A}}
\underbrace{\left[\begin{array}{c}
g_{11}\cos\beta\\ g_{12}\cos\beta\\ g_{13}\cos\beta\\ g_{11}\sin\beta\\ g_{12}\sin\beta\\ g_{13}\sin\beta\\ g_{11}\\ g_{12}\\ g_{13}\\
g_{21}\cos\beta\\ g_{22}\cos\beta\\ g_{23}\cos\beta\\ g_{21}\sin\beta\\ g_{22}\sin\beta\\ g_{23}\sin\beta\\ g_{21}\\ g_{22}\\ g_{23}\\
g_{31}\cos\beta\\ g_{32}\cos\beta\\ g_{33}\cos\beta\\ g_{31}\sin\beta\\ g_{32}\sin\beta\\ g_{33}\sin\beta\\ g_{31}\\ g_{32}\\ g_{33}\\
\hline
\delta B_x\left(g_{11}\underline{k}_{11}+g_{12}\underline{k}_{12}+g_{13}\underline{k}_{13}\right)\\
\delta B_x\left(g_{11}\underline{k}_{12}+g_{12}\underline{k}_{22}+g_{13}\underline{k}_{23}\right)\\
\delta B_x\left(g_{11}\underline{k}_{13}+g_{12}\underline{k}_{23}+g_{13}\underline{k}_{33}\right)\\
\delta B_y\left(g_{11}\underline{k}_{11}+g_{12}\underline{k}_{12}+g_{13}\underline{k}_{13}\right)\\
\delta B_y\left(g_{11}\underline{k}_{12}+g_{12}\underline{k}_{22}+g_{13}\underline{k}_{23}\right)\\
\delta B_y\left(g_{11}\underline{k}_{13}+g_{12}\underline{k}_{23}+g_{13}\underline{k}_{33}\right)\\
\delta B_z\left(g_{11}\underline{k}_{11}+g_{12}\underline{k}_{12}+g_{13}\underline{k}_{13}\right)\\
\delta B_z\left(g_{11}\underline{k}_{12}+g_{12}\underline{k}_{22}+g_{13}\underline{k}_{23}\right)\\
\delta B_z\left(g_{11}\underline{k}_{13}+g_{12}\underline{k}_{23}+g_{13}\underline{k}_{33}\right)\\
\delta B_x\left(g_{21}\underline{k}_{11}+g_{22}\underline{k}_{12}+g_{23}\underline{k}_{13}\right)\\
\delta B_x\left(g_{21}\underline{k}_{12}+g_{22}\underline{k}_{22}+g_{23}\underline{k}_{23}\right)\\
\delta B_x\left(g_{21}\underline{k}_{13}+g_{22}\underline{k}_{23}+g_{23}\underline{k}_{33}\right)\\
\delta B_y\left(g_{21}\underline{k}_{11}+g_{22}\underline{k}_{12}+g_{23}\underline{k}_{13}\right)\\
\delta B_y\left(g_{21}\underline{k}_{12}+g_{22}\underline{k}_{22}+g_{23}\underline{k}_{23}\right)\\
\delta B_y\left(g_{21}\underline{k}_{13}+g_{22}\underline{k}_{23}+g_{23}\underline{k}_{33}\right)\\
\delta B_z\left(g_{21}\underline{k}_{11}+g_{22}\underline{k}_{12}+g_{23}\underline{k}_{13}\right)\\
\delta B_z\left(g_{21}\underline{k}_{12}+g_{22}\underline{k}_{22}+g_{23}\underline{k}_{23}\right)\\
\delta B_z\left(g_{21}\underline{k}_{13}+g_{22}\underline{k}_{23}+g_{23}\underline{k}_{33}\right)\\
\delta B_x\left(g_{31}\underline{k}_{11}+g_{32}\underline{k}_{12}+g_{33}\underline{k}_{13}\right)\\
\delta B_x\left(g_{31}\underline{k}_{12}+g_{32}\underline{k}_{22}+g_{33}\underline{k}_{23}\right)\\
\delta B_x\left(g_{31}\underline{k}_{13}+g_{32}\underline{k}_{23}+g_{33}\underline{k}_{33}\right)\\
\delta B_y\left(g_{31}\underline{k}_{11}+g_{32}\underline{k}_{12}+g_{33}\underline{k}_{13}\right)\\
\delta B_y\left(g_{31}\underline{k}_{12}+g_{32}\underline{k}_{22}+g_{33}\underline{k}_{23}\right)\\
\delta B_y\left(g_{31}\underline{k}_{13}+g_{32}\underline{k}_{23}+g_{33}\underline{k}_{33}\right)\\
\delta B_z\left(g_{31}\underline{k}_{11}+g_{32}\underline{k}_{12}+g_{33}\underline{k}_{13}\right)\\
\delta B_z\left(g_{31}\underline{k}_{12}+g_{32}\underline{k}_{22}+g_{33}\underline{k}_{23}\right)\\
\delta B_z\left(g_{31}\underline{k}_{13}+g_{32}\underline{k}_{23}+g_{33}\underline{k}_{33}\right)\\
\hline
b_x\\ b_y\\ b_z
\end{array}\right]}_{\vec{\theta}^e} \tag{28}$$

Similar to Section 1, $\vec{0}$ is a 9×1 vector of zeros. On the other hand, we now have a larger $\boldsymbol{A}$ with a size of $3N\times 57$. For the observability analysis, we take the same approach and investigate the rank of the linear mapping $\boldsymbol{A}$ and the extended natural mapping $M : \vec{\theta}\in\mathbb{R}^{22}\rightarrow\vec{\theta}^e\in\mathbb{R}^{57}$ where $\vec{\theta}$ is modified as follows assuming that $\boldsymbol{G}$ has nine DoFs and considering the definitions of $\boldsymbol{G}$ and $\underline{\boldsymbol{K}}$:

$$\vec{\theta} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{21} & g_{22} & g_{23} & g_{31} & g_{32} & g_{33} & \underline{k}_{11} & \underline{k}_{12} & \underline{k}_{13} & \underline{k}_{22} & \underline{k}_{23} & \underline{k}_{33} & \beta & \delta B_x & \delta B_y & \delta B_z & b_x & b_y & b_z \end{bmatrix}^T \tag{29}$$

For the first part of the analysis, the rank of $\boldsymbol{A}$ is required to be 22 minimum, which is true in the same way as in Section 1 since $\phi, \gamma,$ and ψ take more than 22 different values in the

$[-\pi, +\pi)$ in our N sample collection. We continue our analysis by deriving the Jacobian $\boldsymbol{J}_M$ of the mapping M. Checking the rank of $\boldsymbol{J}_M$ visually is a little cumbersome this time because of its size. As can be seen easily, the first nine columns and the 16th column of $\boldsymbol{J}_M$ and the last three columns are linearly independent with respect to all columns of the Jacobian. For the remaining nine columns, it is not that easy to say that they are linearly independent as well. To this end, we checked the rank using the symbolic toolbox of MATLAB. As a result, we found out that $\text{rank}(\boldsymbol{J}_\text{M}) = 22$, which leads to the conclusion that the minimization problem $\arg\min_{\vec{\theta}} \|\vec{y} - \boldsymbol{A}\vec{\theta}^e\|$, equivalent to $\arg\min_{\vec{\theta}} \|\vec{y} - \vec{G}(\vec{\theta})\|$ presented in the manuscript, has 22 DoFs. Hence, all of the parameters involved in Equation (29) are observable.

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$$
\boldsymbol{J}_M = \left[\begin{array}{cccccccccccccccccccccc}
\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_{11}\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_{12}\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_{13}\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{11}\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{12}\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{13}\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_{21}\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_{22}\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_{23}\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{21}\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{22}\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{23}\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_{31}\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_{32}\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & -g_{33}\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{31}\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{32}\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & g_{33}\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
\delta B_x\underline{k}_{11} & \delta B_x\underline{k}_{12} & \delta B_x\underline{k}_{13} & 0 & 0 & 0 & 0 & 0 & 0 & \delta B_x g_{11} & \delta B_x g_{12} & \delta B_x g_{13} & 0 & 0 & 0 & 0 & w_{11} & 0 & 0 & 0 & 0 & 0 \\
\delta B_x\underline{k}_{12} & \delta B_x\underline{k}_{22} & \delta B_x\underline{k}_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta B_x g_{11} & 0 & \delta B_x g_{12} & \delta B_x g_{13} & 0 & 0 & w_{12} & 0 & 0 & 0 & 0 & 0 \\
\delta B_x\underline{k}_{13} & \delta B_x\underline{k}_{23} & \delta B_x\underline{k}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta B_x g_{11} & 0 & \delta B_x g_{12} & \delta B_x g_{13} & 0 & w_{13} & 0 & 0 & 0 & 0 & 0 \\
\delta B_y\underline{k}_{11} & \delta B_y\underline{k}_{12} & \delta B_y\underline{k}_{13} & 0 & 0 & 0 & 0 & 0 & 0 & \delta B_y g_{11} & \delta B_y g_{12} & \delta B_y g_{13} & 0 & 0 & 0 & 0 & 0 & w_{11} & 0 & 0 & 0 & 0 \\
\delta B_y\underline{k}_{12} & \delta B_y\underline{k}_{22} & \delta B_y\underline{k}_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta B_y g_{11} & 0 & \delta B_y g_{12} & \delta B_y g_{13} & 0 & 0 & 0 & w_{12} & 0 & 0 & 0 & 0 \\
\delta B_y\underline{k}_{13} & \delta B_y\underline{k}_{23} & \delta B_y\underline{k}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta B_y g_{11} & 0 & \delta B_y g_{12} & \delta B_y g_{13} & 0 & 0 & w_{13} & 0 & 0 & 0 & 0 \\
\delta B_z\underline{k}_{11} & \delta B_z\underline{k}_{12} & \delta B_z\underline{k}_{13} & 0 & 0 & 0 & 0 & 0 & 0 & \delta B_z g_{11} & \delta B_z g_{12} & \delta B_z g_{13} & 0 & 0 & 0 & 0 & 0 & 0 & w_{11} & 0 & 0 & 0 \\
\delta B_z\underline{k}_{12} & \delta B_z\underline{k}_{22} & \delta B_z\underline{k}_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta B_z g_{11} & 0 & \delta B_z g_{12} & \delta B_z g_{13} & 0 & 0 & 0 & 0 & w_{12} & 0 & 0 & 0 \\
\delta B_z\underline{k}_{13} & \delta B_z\underline{k}_{23} & \delta B_z\underline{k}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta B_z g_{11} & 0 & \delta B_z g_{12} & \delta B_z g_{13} & 0 & 0 & 0 & w_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & \delta B_x\underline{k}_{11} & \delta B_x\underline{k}_{12} & \delta B_x\underline{k}_{13} & 0 & 0 & 0 & \delta B_x g_{21} & \delta B_x g_{22} & \delta B_x g_{23} & 0 & 0 & 0 & 0 & w_{21} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta B_x\underline{k}_{12} & \delta B_x\underline{k}_{22} & \delta B_x\underline{k}_{23} & 0 & 0 & 0 & 0 & \delta B_x g_{21} & 0 & \delta B_x g_{22} & \delta B_x g_{23} & 0 & 0 & w_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta B_x\underline{k}_{13} & \delta B_x\underline{k}_{23} & \delta B_x\underline{k}_{33} & 0 & 0 & 0 & 0 & 0 & \delta B_x g_{21} & 0 & \delta B_x g_{22} & \delta B_x g_{23} & 0 & w_{23} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta B_y\underline{k}_{11} & \delta B_y\underline{k}_{12} & \delta B_y\underline{k}_{13} & 0 & 0 & 0 & \delta B_y g_{21} & \delta B_y g_{22} & \delta B_y g_{23} & 0 & 0 & 0 & 0 & 0 & w_{21} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta B_y\underline{k}_{12} & \delta B_y\underline{k}_{22} & \delta B_y\underline{k}_{23} & 0 & 0 & 0 & 0 & \delta B_y g_{21} & 0 & \delta B_y g_{22} & \delta B_y g_{23} & 0 & 0 & 0 & w_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta B_y\underline{k}_{13} & \delta B_y\underline{k}_{23} & \delta B_y\underline{k}_{33} & 0 & 0 & 0 & 0 & 0 & \delta B_y g_{21} & 0 & \delta B_y g_{22} & \delta B_y g_{23} & 0 & 0 & w_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta B_z\underline{k}_{11} & \delta B_z\underline{k}_{12} & \delta B_z\underline{k}_{13} & 0 & 0 & 0 & \delta B_z g_{21} & \delta B_z g_{22} & \delta B_z g_{23} & 0 & 0 & 0 & 0 & 0 & 0 & w_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & \delta B_z\underline{k}_{12} & \delta B_z\underline{k}_{22} & \delta B_z\underline{k}_{23} & 0 & 0 & 0 & 0 & \delta B_z g_{21} & 0 & \delta B_z g_{22} & \delta B_z g_{23} & 0 & 0 & 0 & 0 & w_{22} & 0 & 0 & 0 \\
0 & 0 & 0 & \delta B_z\underline{k}_{13} & \delta B_z\underline{k}_{23} & \delta B_z\underline{k}_{33} & 0 & 0 & 0 & 0 & 0 & \delta B_z g_{21} & 0 & \delta B_z g_{22} & \delta B_z g_{23} & 0 & 0 & 0 & w_{23} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta B_x\underline{k}_{11} & \delta B_x\underline{k}_{12} & \delta B_x\underline{k}_{13} & \delta B_x g_{31} & \delta B_x g_{32} & \delta B_x g_{33} & 0 & 0 & 0 & 0 & w_{31} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta B_x\underline{k}_{12} & \delta B_x\underline{k}_{22} & \delta B_x\underline{k}_{23} & 0 & \delta B_x g_{31} & 0 & \delta B_x g_{32} & \delta B_x g_{33} & 0 & 0 & w_{32} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta B_x\underline{k}_{13} & \delta B_x\underline{k}_{23} & \delta B_x\underline{k}_{33} & 0 & 0 & \delta B_x g_{31} & 0 & \delta B_x g_{32} & \delta B_x g_{33} & 0 & w_{33} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta B_y\underline{k}_{11} & \delta B_y\underline{k}_{12} & \delta B_y\underline{k}_{13} & \delta B_y g_{31} & \delta B_y g_{32} & \delta B_y g_{33} & 0 & 0 & 0 & 0 & 0 & w_{31} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta B_y\underline{k}_{12} & \delta B_x\underline{k}_{22} & \delta B_y\underline{k}_{23} & 0 & \delta B_y g_{31} & 0 & \delta B_y g_{32} & \delta B_y g_{33} & 0 & 0 & 0 & w_{32} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta B_y\underline{k}_{13} & \delta B_y\underline{k}_{23} & \delta B_y\underline{k}_{33} & 0 & 0 & \delta B_y g_{31} & 0 & \delta B_y g_{32} & \delta B_y g_{33} & 0 & 0 & w_{33} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta B_z\underline{k}_{11} & \delta B_z\underline{k}_{12} & \delta B_z\underline{k}_{13} & \delta B_z g_{31} & \delta B_z g_{32} & \delta B_z g_{33} & 0 & 0 & 0 & 0 & 0 & 0 & w_{31} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta B_z\underline{k}_{12} & \delta B_z\underline{k}_{22} & \delta B_z\underline{k}_{23} & 0 & \delta B_z g_{31} & 0 & \delta B_z g_{32} & \delta B_z g_{33} & 0 & 0 & 0 & 0 & w_{32} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta B_z\underline{k}_{13} & \delta B_z\underline{k}_{23} & \delta B_z\underline{k}_{33} & 0 & 0 & \delta B_z g_{31} & 0 & \delta B_z g_{32} & \delta B_z g_{33} & 0 & 0 & 0 & w_{33} & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \tag{30}
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