Nonlinear Systems : Chapter 4

Describing Function Analysis

• Aim : To predict the limit cycles

• Method : Generalization of Frequency domain Methods

• This method has the following advantages :
  
  • i : Results in algebraic, but complex, analysis
  
  • ii : Uses phasor type analysis
  
  • iii : Suitable for graphical interpretation
  
  • iv : Suitable for experimental analysis

• Example : Van der Pol Oscillator

• $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$

• It is well known that this system has a unique limit cycle for large $\mu$

• $\ddot{x} - \mu\dot{x} + x = -\mu x^2 \dot{x}$

• Set $u = -x^2 \dot{x} \implies \ddot{x} - \mu\dot{x} + x = \mu u$

• Take Laplace transform $\implies G(s) = \frac{X(s)}{U(s)} = \frac{\mu}{s^2 - \mu s + 1}$
• Assume that the system is in the limit cycle. **Question**: What is the amplitude and the fundamental frequency of the oscillation?

• Consider the signal in front of the Nonlinear block. Call it $v$. In the present case, $v = -x$

• Set $v = A \sin \omega t \implies u = v^2 \dot{v} = A^2 (\sin \omega t)^2 A \omega \cos \omega t$

• $\sin^2 \theta = 0.5(1 - \cos 2\theta)$, $\cos a \cos b = 0.5[\cos(a + b) + \cos(a - b)]$

• $\implies u = 0.25A^3 \omega (\cos \omega t - \cos 3\omega t)$

• Assume that the linear block acts like a **low pass filter** $\implies$ neglect $3\omega$ frequency

• $v = A \sin \omega t \implies u = 0.25A^3 \omega \cos \omega t$

• Remember Phasor Analysis: $A \cos(\omega t + \phi) = \Re \{A e^{j\phi} e^{j\omega t}\}$

• $y(t) = A \cos(\omega t + \phi) \iff$ Phasor $Y$ of $y(t)$ is $Y = Ae^{j\phi}$

• $v = A \sin \omega t = A \cos(\omega t - 0.5\pi) \iff$ Phasor $V$ is $V = Ae^{-j0.5\phi} = -jA$

• $u = 0.25A^3 \omega \cos \omega t \iff$ Phasor $U$ is $U = 0.25A^3 \omega$
• For **phasor analysis**, the NonLinear (NL) Block can be replaced by Quasi-Linear Block

• The transfer function of this block, \( N(A, \omega) \) is called the **describing function** of the NL block:

\[
N(A, \omega) = \frac{\text{Output Phasor}}{\text{Input Phasor}} = \frac{U}{V} = \frac{0.25A^3\omega}{-jA} = 0.25A^2(j\omega)
\]

\[
U = N(A, \omega)V , \quad X = G(j\omega)U , \quad V = -X
\]

• **Basic Equation**:

\[
(1 + N(A, \omega)G(j\omega))X = 0 \iff 1 + N(A, \omega)G(j\omega) = 0
\]

• Note that this equation is **complex**; Hence we can find the two unknowns: \( A \) (magnitude of the possible oscillation), and \( \omega \) (frequency of the possible oscillation)

• For the Van der Pol oscillator case:

\[
1 + 0.25A^2(j\omega)\frac{\mu}{(j\omega)^2 - \mu(j\omega) + 1} = 0
\]

\[
(1 - \omega^2) + \mu\omega(0.25A^2 - 1) = 0 \implies \omega = 1 , \quad A = 2
\]

• Hence a possible oscillation: \( x(t) = 2 \sin t \)
• Stability Analysis

• Set $j\omega \rightarrow s$ :

• $1 + 0.25A^2 s \frac{\mu}{s^2 - \mu s + 1} = 0 \implies s^2 + \mu(0.25A^2 - 1)s + 1 = 0$

• $s_{1,2} = -0.5\mu(0.25A^2 - 1) \pm \sqrt{0.25\mu^2(0.25A^2 - 1)^2 - 1}$

• If $A \uparrow 2 \implies \Re\{s_{1,2}\} < 0 \implies$ System becomes stable $\implies A \downarrow$

• If $A \downarrow 2 \implies \Re\{s_{1,2}\} > 0 \implies$ System becomes unstable $\implies A \uparrow$

• Conclusion: This oscillation is **stable** (i.e. can be sustained)

• Exactly the same analysis is applied to the following type of systems:
• Typical Procedure

• i : Assume $v = A \sin \omega t$ as input to the NL Block

• ii : Find the output $u$ of the NL Block. Consider only the first harmonic

• iii : Find the describing function $N(A, \omega)$ as the ratio of the output phasor $U$ and input phasor $V$

• iv : Solve the main equation $1 + N(A, \omega)G(j\omega) = 0$, find $A$ and $\omega$

• Typical example :

• $x^{(n)} + a_{n-1}x^{(n)} + \ldots + a_0x + f(x) = 0$

• $\implies$ NL Block : $u = f(v)$, Linear Block : $G(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}$

• This could also be extended to the case where $f$ depends on the derivatives of $x$ as well

• Example : Van der Pol oscillator :

• $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = \ddot{x} - \mu\dot{x} + x + \mu x^2 \dot{x} = 0$

• This could also be extended to the case where $r = r_0 \sin \omega t$ To predict the frequency domain behaviour of the closed loop system.
• **Basic Assumptions** (Simple Version)

• **i**: Only one nonlinear block.
  - For simplification. Can be eliminated.

• **ii**: Nonlinearity is memoryless and time-invariant.
  - First can be relaxed. Second is needed for Nyquist stability analysis.

• **iii**: Only the fundamental frequency at the output is considered.
  - This is a fundamental assumption and cannot be relaxed. This is physically justified if the linear block is a low-pass filter.

• **iv**: Nonlinearity is odd (i.e. \( f(e) = -f(-e) \)).
  - Simplifies the Fourier analysis.

• Assume that the input to the NL block is sinusoidal, i.e. \( v(t) = A \sin \omega t \), and NL Block gives \( u(t) = f(v(t)) \).

• If \( f \) is single valued function, than \( u(t) \) is also periodic with the same fundamental frequency
  - \( u(t) = f(A \sin \omega t) = f(A \sin \omega t + 2n\pi) = f(A \sin \omega(t + 2n\pi/\omega)) \)
  - \( u(t) = u(t + nT) \), \( T = 2n\pi/\omega \)
As a result, we can express $u(t)$ as a Fourier series

$$u(t) = 0.5a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

Remember the orthogonality of sin and cos functions, i.e. : For $n \neq m$

$$\int_{-\pi/\omega}^{\pi/\omega} \sin n\omega t \sin m\omega t \, dt = \int_{-\pi/\omega}^{\pi/\omega} \cos n\omega t \cos m\omega t \, dt$$

$$= \int_{-\pi/\omega}^{\pi/\omega} \sin n\omega t \cos m\omega t \, dt = 0$$

Using this property, we easily obtain:

$$\int_{-\pi/\omega}^{\pi/\omega} u(t) \sin n\omega t \, dt = b_n \int_{-\pi/\omega}^{\pi/\omega} (\sin n\omega t)^2 \, dt$$

$$= \int_{-\pi/\omega}^{\pi/\omega} 0.5(1 - \cos 2n\omega t) \, dt = b_n \pi/\omega$$

$$b_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} u(t) \sin n\omega t \, dt$$

Coordinate change : $\omega t = \theta$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(A \sin \theta) \sin n\theta \, d\theta$$

Similarly we have:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(A \sin \theta) \cos n\theta \, d\theta$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(A \sin \theta) \, d\theta$$
• If $f$ is odd $\implies a_0 = 0!$

• Fundamental terms $\implies u(t) = a_1 \cos \omega t + b_1 \sin \omega t$

• Phasor of $b_1 \sin \omega t \implies -jb_1$

• Phasor of $u(t) \implies a_1 - jb_1$

• Describing Function :

• $N(A, \omega) = \frac{\text{Output Phasor}}{\text{Input Phasor}} = \frac{U}{V} = \frac{a_1 - jb_1}{-jA} = \frac{b_1 + ja_1}{A}$