

On the Optimality of Equal Gain Combining for Energy Detection of Unknown Signals

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Abstract—In this letter, the optimality of equal gain combining (EGC) for energy detection of unknown signals in a system with receive-diversity is studied. It is shown that the EGC results in an optimal test statistic for infinitesimally small signal-to-noise ratios (SNRs); however, it does not yield the optimal test for all SNR values according to the Neyman-Pearson criterion. In other words, EGC does not induce a uniformly most powerful (UMP) test for unknown signal detection.

I. INTRODUCTION

Detecting the presence of unknown signals [1] has many applications in communications and radar problems, such as demodulation of on-off keyed signals, coarse signal acquisition and aircraft detection [2]-[3]. Commonly, energy detectors are employed in order to detect the presence of unknown signals. An energy detector measures the energy of an incoming signal and compares it to an appropriately selected threshold to determine the presence of a transmitted signal.

For an additive white Gaussian noise (AWGN) channel model, the energy of the signal can be modeled by central and non-central chi-square distributed random variables, respectively, for noise-only and signal-plus-noise hypotheses [4]. Under this model, it is well-known that comparing the energy level to a threshold corresponding to the maximum allowable false-alarm (FA) rate is the uniformly most powerful (UMP) test for signal detection [5].

When there is receive-diversity in the system, one can obtain a set of energy measurements from a number of diversity branches. For example, pulses are transmitted over a number of time intervals in an impulse radio (IR) ultra-wideband (UWB) acquisition problem; or, a transmitted signal can be received by a number of antenna elements in a multiple-output system. In such cases, the conventional approach is to employ equal gain combining (EGC) for the set of measurements, and compare the combined output to a threshold [6], [7]. Although comparing the energy of a single measurement to a threshold is a UMP test, no studies have been performed to analyze the optimality of the test that compares the EGC of a given set of energy measurements (from a set of diversity branches) to a threshold.

In this letter, we show that EGC results in a locally most powerful (LMP) test for unknown signal detection. More specifically, for detecting unknown signals, comparing the total energy of a set of diversity branches to a threshold

gives the most powerful test for infinitesimally small signal-to-noise ratios (SNRs). However, we also show that this test is not uniformly most powerful (UMP) for signal detection. In other words, if the SNR is not infinitesimally small, the EGC becomes suboptimal. By showing that the EGC results in an LMP test, but not a UMP test, we also implicitly prove that there is not a UMP test for the detection of an unknown signal given a set of energy measurements from a set of diversity branches.

II. SIGNAL MODEL

Consider an energy detection problem in which K observations are obtained as follows:

$$y_i = \int |r_i(t)|^2 dt, \quad (1)$$

for $i = 1, \dots, K$, where $r_i(t)$ denotes the received signal for the i th observation. For example, $r_i(t)$ can be the received signal at the i th antenna for a system with K receive antennas, or it can be the received signal corresponding to the i th time interval for a system with repetition diversity³.

Depending on whether a signal is present in the observations, y_i in (1) can be expressed as

$$y_i = \int |n_i(t)|^2 dt, \quad (2)$$

for $i = 1, \dots, K$, for the noise-only hypothesis (\mathcal{H}_0), where $n_i(t)$ is the zero-mean Gaussian noise with a flat spectral density of σ_n^2 over the system bandwidth, and is independent for different observations, and

$$y_i = \int |s_i(t) + n_i(t)|^2 dt, \quad (3)$$

for $i = 1, \dots, K$, for the signal-plus-noise hypothesis (\mathcal{H}_1), where $s_i(t)$ is the unknown signal in the i th observation.

The problem is to test the noise-only hypothesis against a signal-plus-noise hypothesis, given K observations in (1). From [4], we can show that y_i has a central or non-central chi-square distribution, respectively, for the noise-only and signal-plus-noise hypotheses:

$$\begin{aligned} \mathcal{H}_0 : y_i &\sim \chi_M^2(0) & i = 1, \dots, K, \\ \mathcal{H}_1 : y_i &\sim \chi_M^2(\theta_i) & i = 1, \dots, K, \end{aligned} \quad (4)$$

where M is the approximate dimensionality of the signal space, which is obtained from the time-bandwidth product⁴

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³A UWB system can be a good example for this, in which a number of pulses are transmitted per information symbol.

⁴Note that a common M can be used in (4) without loss of generality, assuming that it is selected according to the signal with the largest time-bandwidth product.

[4], θ_i is the energy of the signal for the i th observation, $\theta_i = \int |s_i(t)|^2 dt$, and $\chi_M^2(\theta)$ denotes a non-central chi-square distribution with M degrees of freedom and a non-centrality parameter of θ . Clearly, $\chi_M^2(\theta)$ reduces to a central chi-square distribution with M degrees of freedom for $\theta = 0$.

We start with the assumption that the signal has the same energy in different observations; that is, $\theta_i = \theta \forall i$. Then, we extend the results to the more general case in Section IV. Note that $\theta_i = \theta \forall i$ can be valid in certain scenarios; for example, in a system employing repetition diversity in a slow-fading environment, the signal energy is the same for different observations, as can be observed in a UWB acquisition problem.

III. LOCALLY AND UNIFORMLY MOST POWERFUL TESTS

It is observed from (4) that the energy detection of unknown signals is a composite hypothesis testing problem, which compares the simple hypothesis $\theta = 0$ against a composite hypothesis $\theta > 0$ (assuming $\theta_i = \theta \forall i$) [5]. In general, it is difficult to assign prior probabilities to these hypotheses in practical situations; especially, for acquisition, time-of-arrival and radar problems. Therefore, we do not assume any priors for the signal presence, and analyze the optimality of the conventional EGC scheme in the Neyman-Pearson framework⁵ [5].

We first focus on the case in which θ is close to zero, which corresponds to the problem of detecting weak signals. This can be realistic in certain scenarios, such as for detecting UWB signals, which usually have low SNRs. In such a case, the following proposition shows that the EGC results in the most powerful test; i.e. comparing the total signal energy to a threshold maximizes the detection probability for a given FA rate.

Proposition 1: *Given an FA rate α and the energy samples from K observations of the received signal, y_1, \dots, y_K , the LMP test for signal detection compares the total energy to a threshold as follows:*

$$\sum_{i=1}^K y_i \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} h^{-1}(\alpha), \quad (5)$$

where h is a monotone decreasing function expressed as

$$h(x) = e^{-\frac{x}{2\sigma_n^2}} \sum_{k=0}^{MK-1} \frac{1}{k!} \left(\frac{x}{2\sigma_n^2} \right)^k. \quad (6)$$

Proof: From (4) and the independent and identically distributed (i.i.d.) nature of $\{y_i\}_{i=1}^K$ in this constant-signal-energy case, the likelihood ratio can be expressed as

$$L_\theta(\mathbf{y}) = \prod_{i=1}^K \frac{p_\theta(y_i)}{p_0(y_i)}, \quad (7)$$

where $\mathbf{y} = [y_1 \cdots y_K]$, and

$$p_\theta(y) = \frac{1}{2\sigma_n^2} \left(\frac{y}{\theta} \right)^{\frac{M}{4} - \frac{1}{2}} e^{-\frac{(\theta+y)}{2\sigma_n^2}} I_{M/2-1} \left(\frac{\sqrt{y\theta}}{\sigma_n^2} \right), \quad (8)$$

$$p_0(y) = \frac{y^{\frac{M}{2}-1} e^{-\frac{y}{2\sigma_n^2}}}{\sigma_n^M 2^{M/2} \Gamma(M/2)}, \quad (9)$$

with $\Gamma(\cdot)$ being the gamma function [9], and

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)}, \quad (10)$$

for $x \geq 0$, the ν th-order modified Bessel function of the first kind.

After some manipulation, the likelihood ratio can be obtained as

$$L_\theta(\mathbf{y}) = (f(\theta))^K \prod_{i=1}^K g_\theta(y_i), \quad (11)$$

where

$$f(\theta) = \sigma_n^{M-2} 2^{\frac{M}{2}-1} \Gamma(M/2) e^{-\theta/(2\sigma_n^2)}, \quad (12)$$

$$g_\theta(y_i) = \sum_{k=0}^{\infty} l_k \theta y_i^k, \quad (13)$$

with

$$l_k = [(2\sigma_n^2)^{M/2+2k-1} k! \Gamma(M/2+k)]^{-1}. \quad (14)$$

For the LMP test, the decision rule [5],

$$\frac{d}{d\theta} L_\theta(\mathbf{y})|_{\theta=0} \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \eta, \quad (15)$$

can be calculated, from (11)-(13), as

$$\sum_{i=1}^K y_i \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \tilde{\eta}, \quad (16)$$

where $\tilde{\eta} = (2\sigma_n^2 \eta + K) 2\sigma_n^2 \Gamma(m/2+1) / \Gamma(m/2)$.

In order to determine the threshold $\tilde{\eta}$, we consider the FA constraint:

$$\alpha = \text{P} \left\{ \sum_{i=1}^K y_i > \tilde{\eta} | \mathcal{H}_0 \right\} = e^{-\tilde{\eta}/(2\sigma_n^2)} \sum_{k=0}^{KM-1} \frac{1}{k!} \left(\frac{\tilde{\eta}}{2\sigma_n^2} \right)^k = h(\tilde{\eta}), \quad (17)$$

where the second equality is obtained by using the expression for the cumulative distribution function (CDF) of a central chi-square distributed random variable with KM degrees of freedom [9], since the sum of K i.i.d. central chi-square random variables with M degrees of freedom results in another central chi-square random variable with KM degrees of freedom.

Since $h(\tilde{\eta})$ in (17) is a monotone decreasing function, a unique $\tilde{\eta}$ can be obtained for a given FA rate α . Hence, the LMP test can be expressed as in the proposition. \square

Note that the proposition states that the comparison of the total block energy to a threshold is an LMP test for signal detection as long as the threshold is selected so that the FA rate is equal to α , the maximum allowable level. In other words, the EGC of the observations is optimal in the Neyman-Pearson sense for infinitesimally small SNRs.

⁵Alternatively, the maximum likelihood criterion can also be considered [8].

Although the total energy test in (5) is the most powerful test for infinitesimally small values of θ , we show in the following proposition that it is not the most powerful test for all θ when $K \geq 2$.

Proposition 2: For $K \geq 2$, the total energy test in (5) is not a UMP test for (4).

Proof: From the likelihood ratio expression in (11), we can express the likelihood ratio test as

$$\prod_{i=1}^K g_{\theta}(y_i) \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\geq}} \delta, \quad (18)$$

where δ is a threshold, which is set according to the FA rate criterion. Note that the first term in the right-hand-side of (11) is implicitly included in the threshold of the test in (18).

We know that a UMP test exists for (4) if and only if the critical region for a given FA rate, the set of observations satisfying $\prod_{i=1}^K g_{\theta}(y_i) > \delta_{\theta}$, can be made independent of the value of θ for all $\theta > 0$ [5]. In order for the total energy test to be UMP, we should be able to reduce that critical region to the set of observations satisfying $\sum_{i=1}^K y_i > \tilde{\delta}$. If we can prove that this is impossible, then the proof of the proposition will be complete⁶. In order to prove this, we can check the boundary of the critical region, $\prod_{i=1}^K g_{\theta}(y_i) = \delta_{\theta}$. If the boundary of the critical region is not linear, then the critical region is different from that of the total energy test, which proves the claim in the proposition.

First, consider $K = 2$. In this case, the slope of the boundary should be equal to -1 so that it can be expressed as $y_1 + y_2 = \tilde{\delta}$ for an appropriate $\tilde{\delta}$. We can show that the slope condition is satisfied if

$$\frac{g_{\theta}(y_1)}{g'_{\theta}(y_1)} = \frac{g_{\theta}(y_2)}{g'_{\theta}(y_2)}, \quad (19)$$

for all y_1 and y_2 , where $g'_{\theta}(y_i)$ represents the derivative of $g_{\theta}(y_i)$ with respect to y_i .

We can prove that (19) is not true by simply evaluating it at a certain point. For example, for $y_1 = 0$, $y_2 = \tilde{\delta}$ and $\theta = 1$, (19) reduces, using (13), to

$$\frac{l_1}{l_0} = \frac{2l_2 + 3l_3\tilde{\delta} + 4l_4\tilde{\delta}^2 + \dots}{l_1 + l_2\tilde{\delta} + l_3\tilde{\delta}^2 + \dots}. \quad (20)$$

From (14), (20) can be expressed, after some manipulation, as

$$1 = \frac{\frac{M}{2} \left(2l_2 + 3l_3\tilde{\delta} + 4l_4\tilde{\delta}^2 + \dots \right)}{\left(\frac{M}{2} + 1 \right) 2l_2 + \left(\frac{M}{2} + 2 \right) 3l_3\tilde{\delta} + \left(\frac{M}{2} + 3 \right) 4l_4\tilde{\delta}^2 + \dots},$$

which is obviously false since the denominator is strictly larger than the numerator.

Therefore, we see that the slope of the boundary is not -1 . In fact, it is easy to see that the boundary is not linear since the slope is not constant. For example, for $y_1 = y_2 = \tilde{\delta}/2$, (19) is satisfied; hence, the slope is -1 . Therefore, the slope is different at $(0, \tilde{\delta})$ and $(\tilde{\delta}/2, \tilde{\delta}/2)$, which shows that the boundary is not linear.

⁶For $K = 1$, the the critical region $g_{\theta}(y_1) > \delta$ reduces to $y_1 > \tilde{\delta}$ since $g_{\theta}(y_1)$ is a monotone increasing function of y_1 . Hence, comparing y_1 to a threshold is a UMP test in this case.

For the $K > 2$ case, it follows from gradient arguments that in order for the total energy test to be UMP, the following equalities should be satisfied:

$$\frac{g_{\theta}(y_1)}{g'_{\theta}(y_1)} = \dots = \frac{g_{\theta}(y_K)}{g'_{\theta}(y_K)}, \quad (21)$$

for all y_1, \dots, y_K . From the previous analysis, it directly follows that (21) is not true, either.

Hence, we conclude that the total energy test is not a UMP test for the problem in (4). \square

IV. CONCLUSIONS AND EXTENSIONS

We can deduce from Propositions 1 and 2 that there is no UMP test for the problem in (4). This is because, if there were one, it would be the total energy test since it is already the most powerful test around $\theta = 0$.

Moreover, we observe that the EGC is optimal for infinitesimally small θ when the threshold is selected appropriately according to the given FA rate. However, it is not optimal in general for any given θ . In other words, given a set of energy measurements from a number of diversity branches, the EGC is not always optimal.

Finally, we can extend the results to the case in which the signal energy θ_i in (4) is not constant for all observations. In this case, it can be shown that the result of Proposition 1 is still true, since

$$\frac{\partial}{\partial \theta_i} L_{\theta}(\mathbf{y})|_{\theta=0} = c_1 \sum_{k=1}^K y_k - c_2, \quad (22)$$

for $i = 1, \dots, K$, where $\theta = [\theta_1 \dots \theta_K]$, and c_1 and c_2 are positive constants. Also, it is straightforward to show that the EGC does not result in a UMP test, since it is already not optimal for the constant energy, $\theta_i = \theta \forall i$, case. In other words, in the most general case, the test is not UMP for the alternative $[0, \infty]^K$, because it is not UMP on the diagonal subset of this alternative.

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