On the Improvability and Non-improvability of Detection via Additional Independent Noise

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***Abstract*— Addition of independent noise to measurements can improve performance of some suboptimal detectors under certain conditions. In this letter, sufficient conditions under which the performance of a suboptimal detector cannot be enhanced by additional independent noise are derived according to the Neyman-Pearson criterion. Also, sufficient conditions are obtained to specify when the detector performance can be improved. In addition to a generic condition, various explicit sufficient conditions are proposed for easy evaluation of improvability. Finally, a numerical example is presented and the practicality of the proposed conditions is discussed.**

***Index Terms*— Detection, binary hypothesis-testing, Neyman-Pearson.**

I. Introduction

Performance of some suboptimal detectors can be improved by adding independent noise to their measurements. Improving the performance of a detector by adding a stochastic signal to the measurement can be considered in the framework of *stochastic resonance* (SR), which can be regarded as the observation of *noise benefits* related to signal transmission in nonlinear systems (please refer to [1]-[5] and references therein for a detailed review of SR). In other words, for some detectors, addition of controlled "noise" can improve detection performance. Such noise benefits can be in various forms, such as an increase in output signal-to-noise ratio (SNR) [2], [6], a decrease in probability of error [7], or an increase in probability of detection under a false-alarm rate constraint [1], [8].

In this study, noise benefits are investigated in the Neyman-Pearson framework [1], [8]; that is, improvements in the probability of detection are considered under a constraint on the probability of false-alarm. In [8], a theoretical framework is developed for this problem, and the probability density function (PDF) of optimal additional noise is specified. Specifically, it is proven that optimal noise can be characterized by a randomization of at most two discrete signals. Moreover, [8] provides sufficient conditions under which the performance of a suboptimal detector can or cannot be improved via additional independent noise. The study in [1] focuses on the same problem and obtains the optimal additional noise PDF via an optimization theoretic approach. In addition, it derives alternative improvability conditions for the case of scalar observations.

In this paper, new improvability and non-improvability conditions are proposed for detectors in the Neyman-Pearson framework, and the improvability conditions in [1] are extended. The results also provide alternative sufficient conditions to those in [8]. In other words, new sufficient conditions are derived, under which the detection probability of a suboptimal detector can or cannot be improved by additional independent noise, under a constraint on the probability of false alarm. All the proposed conditions are defined in terms of the probabilities of detection and false alarm for given additional noise values (cf. (5)) without the need for any other auxiliary functions employed in [8]. In addition to deriving generic conditions, simpler but less generic improvability conditions are provided for practical purposes. The results are compared to those in [8], and the advantages and disadvantages are specified for both approaches. In other words, comments are provided regarding specific detection problems, for which one approach can be more suitable than the other. Moreover, the improvability conditions in [1] for scalar observations are extended to more generic conditions for the case of vector observations. Finally, a numerical example is presented to illustrate an application of the improvability results.

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II. Signal Model

Consider a binary hypothesis-testing problem described as

$$\mathcal{H}_0 \;:\; p_0(\mathbf{x}) \;, \quad \mathcal{H}_1 \;:\; p_1(\mathbf{x}) \;, \tag{1}$$

where $\mathbf{x}$ is the K-dimensional data (measurement) vector, and $p_0(\mathbf{x})$ and $p_1(\mathbf{x})$ represent the PDFs of $\mathbf{x}$ under $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively.

The decision rule (detector) is denoted by $\phi(\mathbf{x})$, which maps the data vector into a real number in $[0, 1]$, representing the probability of selecting $\mathcal{H}_1$ [9]. Under certain circumstances, detector performance can be improved by adding independent noise to the data vector $\mathbf{x}$ [1], [8]. Let $\mathbf{y}$ represent the modified data vector given by $\mathbf{y} = \mathbf{x} + \mathbf{n}$, where $\mathbf{n}$ represents the additional independent noise term.

The Neyman-Pearson framework is considered in this study, and performance of a detector is specified by its probability of detection and probability of false alarm [9]. Since the additional noise is independent of the data, the probabilities of detection and false alarm are given, respectively, by

$$\mathrm{P}_{\mathrm{D}}^{\mathbf{y}} = \int_{\mathbb{R}^K} \phi(\mathbf{y}) \left[\int_{\mathbb{R}^K} p_1(\mathbf{y}-\mathbf{x}) p_{\mathbf{N}}(\mathbf{x}) d\mathbf{x} \right] d\mathbf{y} \;, \tag{2}$$

$$\mathrm{P}_{\mathrm{F}}^{\mathbf{y}} = \int_{\mathbb{R}^K} \phi(\mathbf{y}) \left[\int_{\mathbb{R}^K} p_0(\mathbf{y}-\mathbf{x}) p_{\mathbf{N}}(\mathbf{x}) d\mathbf{x} \right] d\mathbf{y} \;, \tag{3}$$

where K is the dimension of the data vector. After some manipulation, (2) and (3) can be expressed as [8]

$$\mathrm{P}_{\mathrm{D}}^{\mathbf{y}} = \mathrm{E}\{F_1(\mathbf{N})\} \;, \quad \mathrm{P}_{\mathrm{F}}^{\mathbf{y}} = \mathrm{E}\{F_0(\mathbf{N})\} \;, \tag{4}$$

where $\mathbf{N}$ is the random variable representing the additional noise term and

$$F_i(\mathbf{n}) \doteq \int_{\mathbb{R}^K} \phi(\mathbf{y}) p_i(\mathbf{y}-\mathbf{n}) d\mathbf{y} \;, \quad i = 0, 1 \;. \tag{5}$$

Note that in the absence of additional noise, i.e., $\mathbf{n} = \mathbf{0}$, the probabilities of detection and false alarm are given by $\mathrm{P}_{\mathrm{D}}^{\mathbf{x}} = F_1(\mathbf{0})$ and $\mathrm{P}_{\mathrm{F}}^{\mathbf{x}} = F_0(\mathbf{0})$, respectively. The detector $\phi(\cdot)$ is called *improvable* if there exists additional noise $\mathbf{n}$ that satisfies $\mathrm{P}_{\mathrm{D}}^{\mathbf{y}} > \mathrm{P}_{\mathrm{D}}^{\mathbf{x}} = F_1(\mathbf{0})$ and $\mathrm{P}_{\mathrm{F}}^{\mathbf{y}} \le \mathrm{P}_{\mathrm{F}}^{\mathbf{x}} = F_0(\mathbf{0})$. Otherwise, the detector is called *non-improvable*.

III. Non-improvability Conditions

In [8], sufficient conditions for improvability and non-improvability are derived based on the following function:

$$J(t) = \sup\left\{ F_1(\mathbf{n}) \mid F_0(\mathbf{n}) = t \;, \; \mathbf{n} \in \mathbb{R}^K \right\} \;, \tag{6}$$

which defines the maximum probability of detection, obtained by adding constant noise $\mathbf{n}$, for a given probability of false alarm. It is stated that if there exists a non-decreasing concave function $\Psi(t)$ that satisfies $\Psi(t) \geq J(t)\ \forall t$ and $\Psi(\mathrm{P}_\mathrm{F}^{\mathbf{x}}) = J(\mathrm{P}_\mathrm{F}^{\mathbf{x}}) = F_1(\mathbf{0})$, then the detector is non-improvable [8]. The main advantage of this result is that it is based on single-variable functions $J(t)$ and $\Psi(t)$ irrespective of the dimension of the data vector. However, in certain cases, it may be difficult to calculate $J(t)$ in (6) or to obtain $\Psi(t)$. Therefore, we aim to derive a non-improvability condition that depends directly on F_0 and F_1 in (5). The following proposition provides a sufficient condition for non-improvability based on convexity and concavity arguments for F_0 and F_1.

Proposition 1: *Assume that $F_0(\mathbf{n}) \leq F_0(\mathbf{0})$ implies $F_1(\mathbf{n}) \leq F_1(\mathbf{0})$ for all $\mathbf{n} \in \mathcal{S}_\mathrm{n}$, where $\mathcal{S}_\mathrm{n}$ is a convex set*[1] *consisting of all possible values of additional noise $\mathbf{n}$. If $F_0(\mathbf{n})$ is a convex function and $F_1(\mathbf{n})$ is a concave function over $\mathcal{S}_\mathrm{n}$, then the detector is non-improvable.*

Proof: Due to the convexity of F_0, the probability of false alarm in (4) can be bounded, via the Jensen's inequality, as

$$\mathrm{P}_\mathrm{F}^{\mathbf{y}} = \mathrm{E}\{F_0(\mathbf{N})\} \geq F_0\left(\mathrm{E}\{\mathbf{N}\}\right)\ . \tag{7}$$

Since $\mathrm{P}_\mathrm{F}^{\mathbf{y}} \leq \mathrm{P}_\mathrm{F}^{\mathbf{x}} = F_0(\mathbf{0})$ is a necessary condition for improvability, (7) implies that $F_0\left(\mathrm{E}\{\mathbf{N}\}\right) \leq F_0(\mathbf{0})$ is required. Since $\mathrm{E}\{\mathbf{N}\} \in \mathcal{S}_\mathrm{n}$, $F_0\left(\mathrm{E}\{\mathbf{N}\}\right) \leq F_0(\mathbf{0})$ implies that $F_1\left(\mathrm{E}\{\mathbf{N}\}\right) \leq F_1(\mathbf{0})$ due to the assumption in the proposition. Therefore,

$$\mathrm{P}_\mathrm{D}^{\mathbf{y}} = \mathrm{E}\{F_1(\mathbf{N})\} \leq F_1\left(\mathrm{E}\{\mathbf{N}\}\right) \leq F_1(\mathbf{0})\ , \tag{8}$$

where the first inequality results from the concavity of F_1. Then, from (7) and (8), it is concluded that $\mathrm{P}_\mathrm{F}^{\mathbf{y}} \leq F_0(\mathbf{0}) = \mathrm{P}_\mathrm{F}^{\mathbf{x}}$ implies $\mathrm{P}_\mathrm{D}^{\mathbf{y}} \leq F_1(\mathbf{0}) = \mathrm{P}_\mathrm{D}^{\mathbf{x}}$. Therefore, the detector is non-improvable.[2] $\square$

Consider the assumption in the proposition, which states that $F_0(\mathbf{n}) \leq F_0(\mathbf{0})$ implies $F_1(\mathbf{n}) \leq F_1(\mathbf{0})$ for all possible values of $\mathbf{n}$. This assumption is realistic in most practical scenarios, since decreasing the probability of false alarm by using a constant additional noise $\mathbf{n}$ does not usually result in an increase in the probability of detection. In fact, if there exists a noise component $\tilde{\mathbf{n}}$ such that $F_0(\tilde{\mathbf{n}}) \leq F_0(\mathbf{0})$ and $F_1(\tilde{\mathbf{n}}) > F_1(\mathbf{0})$, the detector can be improved simply by adding $\tilde{\mathbf{n}}$ to the original data, i.e., for $p_\mathbf{N}(\mathbf{x}) = \delta(\mathbf{x} - \tilde{\mathbf{n}})$. Therefore, the assumption in the proposition is in fact a necessary condition for non-improvability.

As an example application of Proposition 1, consider a hypothesis-testing problem in which $\mathcal{H}_0$ is represented by a zero-mean Gaussian distribution with variance σ^2 and $\mathcal{H}_1$ by a Gaussian distribution with mean $\mu > 0$ and variance σ^2. The decision rule selects $\mathcal{H}_1$ if $y \geq 0.5\mu$ and $\mathcal{H}_0$ otherwise. Let $\mathcal{S}_\mathrm{n} = (-0.5\mu, 0.5\mu)$ represent the set of additional noise values for possible performance improvement. From (5), F_0 and F_1 can be obtained as $F_0(x) = Q\left(\frac{0.5\mu - x}{\sigma}\right)$ and $F_1(x) = Q\left(\frac{-0.5\mu - x}{\sigma}\right)$. It is observed that F_0 is convex and F_1 is concave over $\mathcal{S}_\mathrm{n}$. Therefore, Proposition 1 implies that the detector is non-improvable. Comparison of two Gaussian hypotheses with different means as in this example is encountered, for instance, in signal acquisition problems, where the aim is to determine the presence of a signal component under Gaussian noise for the purpose of aligning an incoming signal with respect to a local reference signal at the receiver [11].

Comparison of the non-improvability condition in Proposition 1 with that in [8], stated at the beginning of this section, reveals that the former provides a more direct way of evaluating the non-improvability since there is no need to obtain auxiliary functions, such as $\Psi(t)$ and $J(t)$ in (6). However, if $J(t)$ can be obtained easily, then the result in [8] can be more advantageous since it always deals with a function of a single variable irrespective of the dimension of the data vector. Therefore, for multi-dimensional measurements, the result in [8] can be preferred if the calculation of $J(t)$ in (6) is tractable.

IV. Improvability Conditions

Based on the definition in (6), it is stated in [8] that the detector is improvable if $J(\mathrm{P}_\mathrm{F}^{\mathbf{x}}) > \mathrm{P}_\mathrm{D}^{\mathbf{x}}$ or $J^{''}(\mathrm{P}_\mathrm{F}^{\mathbf{x}}) > 0$ when $J(t)$ is second-order continuously differentiable around $\mathrm{P}_\mathrm{F}^{\mathbf{x}}$.[3] Similar to the previous section, the aim is to obtain improvability conditions that directly depend on F_0 and F_1 in (5) instead of J in (6).

First, it can be observed from (4) that if there exists a noise component $\tilde{\mathbf{n}}$ such that $F_1(\tilde{\mathbf{n}}) > F_1(\mathbf{0})$ and $F_0(\tilde{\mathbf{n}}) \leq F_0(\mathbf{0})$, then the detector can be improved by using $p_\mathbf{N}(\mathbf{x}) = \delta(\mathbf{x} - \tilde{\mathbf{n}})$. From (6), it is concluded that this result provides a generalization of the $J(\mathrm{P}_\mathrm{F}^{\mathbf{x}}) > \mathrm{P}_\mathrm{D}^{\mathbf{x}}$ condition [8].

In practical scenarios, $F_0(\mathbf{n}) \leq F_0(\mathbf{0})$ commonly implies $F_1(\mathbf{n}) \leq F_1(\mathbf{0})$. Therefore, the previous result cannot be applied in many cases. Hence, a more generic improvability condition is presented in the following proposition.

Proposition 2: *The detector is improvable if there exist $\mathbf{n}_1$ and $\mathbf{n}_2$ that satisfy*

$$\frac{[F_0(\mathbf{0}) - F_0(\mathbf{n}_2)][F_1(\mathbf{n}_1) - F_1(\mathbf{n}_2)]}{F_0(\mathbf{n}_1) - F_0(\mathbf{n}_2)} > F_1(\mathbf{0}) - F_1(\mathbf{n}_2)\,. \tag{9}$$

Proof: Consider additional noise $\mathbf{n}$ with $p_\mathbf{N}(\mathbf{x}) = \lambda\,\delta(\mathbf{x} - \mathbf{n}_1) + (1-\lambda)\,\delta(\mathbf{x} - \mathbf{n}_2)$. The detector is improvable if $\mathbf{n}_1$, $\mathbf{n}_2$, and $\lambda \in [0, 1]$ satisfy

$$\mathrm{P}_\mathrm{F}^{\mathbf{y}} = \mathrm{E}_\mathbf{n}\{F_0(\mathbf{n})\} = \lambda F_0(\mathbf{n}_1) + (1-\lambda) F_0(\mathbf{n}_2) \leq F_0(\mathbf{0}) \tag{10}$$

$$\mathrm{P}_\mathrm{D}^{\mathbf{y}} = \mathrm{E}_\mathbf{n}\{F_1(\mathbf{n})\} = \lambda F_1(\mathbf{n}_1) + (1-\lambda) F_1(\mathbf{n}_2) > F_1(\mathbf{0}) \tag{11}$$

Although $\mathrm{P}_\mathrm{F}^{\mathbf{y}} \leq F_0(\mathbf{0})$ is sufficient for improvability, the equality condition in (10), i.e., $\mathrm{P}_\mathrm{F}^{\mathbf{y}} = F_0(\mathbf{0})$, is satisfied in most practical cases. As studied in Theorem 4 in [8], $\mathrm{P}_\mathrm{F}^{\mathbf{y}} < F_0(\mathbf{0})$ implies a trivial case in which the detector can be improved by using a constant noise value. Therefore, the equality condition in (10) can be considered, although it is not a necessary condition. Then, λ can be expressed as $\lambda = [F_0(\mathbf{0}) - F_0(\mathbf{n}_2)]/[F_0(\mathbf{n}_1) - F_0(\mathbf{n}_2)]$, which can be inserted in (11) to obtain (9). $\square$

Although the condition in Proposition 2 can directly be evaluated based on F_0 and F_1 functions in (5), finding suitable $\mathbf{n}_1$ and $\mathbf{n}_2$ values can be time consuming in some cases. In fact, it may not always be simpler to check the condition in Proposition 2 than to calculate the optimal noise PDF as in [8]. Therefore, more explicit and simpler improvability conditions are derived in the following.

Proposition 3: *Assume that $F_0(\mathbf{x})$ and $F_1(\mathbf{x})$ are second-order continuously differentiable around $\mathbf{x} = \mathbf{0}$. Define $f_j^{(1)}(\mathbf{x}, \mathbf{z}) \doteq \sum_{i=1}^{K} z_i \frac{\partial F_j(\mathbf{x})}{\partial x_i}$ and $f_j^{(2)}(\mathbf{x}, \mathbf{z}) \doteq \sum_{l=1}^{K} \sum_{i=1}^{K} z_l z_i \frac{\partial^2 F_j(\mathbf{x})}{\partial x_l \partial x_i}$ for $j = 0, 1$, where x_i and z_i represent the ith components of $\mathbf{x}$ and $\mathbf{z}$, respectively. The detector*

[1] Since convex combination of individual noise components can be obtained via randomization [10], $\mathcal{S}_\mathrm{n}$ can be modeled as convex.

[2] It would be sufficient to perform the proof for discrete PDFs, since it is shown in [1] and [8] that the optimal noise PDF is in the form of $p_\mathbf{N}(\mathbf{x}) = \lambda\,\delta(\mathbf{x} - \mathbf{n}_1) + (1-\lambda)\delta(\mathbf{x} - \mathbf{n}_2)$.

[3] In this paper, $J^{'}(a)$ and $J^{''}(a)$ are used to represent, respectively, the first and second derivatives of $J(t)$ at $t = a$.

is improvable if there exists a K-dimensional vector $\mathbf{z}$ such that $f_j^{(1)}(\mathbf{x},\mathbf{z}) > 0$ for $j = 0, 1$ and

$$f_1^{(2)}(\mathbf{x},\mathbf{z}) f_0^{(1)}(\mathbf{x},\mathbf{z}) > f_0^{(2)}(\mathbf{x},\mathbf{z}) f_1^{(1)}(\mathbf{x},\mathbf{z}) \tag{12}$$

are satisfied at $\mathbf{x} = \mathbf{0}$.

Proof: Consider the improvability conditions in (10) and (11) with infinitesimally small noise components, $\mathbf{n}_j = \boldsymbol{\epsilon}_j$ for $j = 1, 2$. Then, $F_i(\boldsymbol{\epsilon}_j)$ can be approximated by using the Taylor series expansion as $F_i(\mathbf{0}) + \boldsymbol{\epsilon}_j^T \mathbf{f}_i + 0.5\, \boldsymbol{\epsilon}_j^T \mathbf{H}_i \boldsymbol{\epsilon}_j$, where $\mathbf{H}_i$ and $\mathbf{f}_i$ are the Hessian and the gradient of $F_i(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$, respectively. Therefore, (10) and (11) require

$$\begin{aligned}
&\lambda\, \boldsymbol{\epsilon}_1^T \mathbf{H}_0 \boldsymbol{\epsilon}_1 + (1-\lambda) \boldsymbol{\epsilon}_2^T \mathbf{H}_0 \boldsymbol{\epsilon}_2 + 2[\lambda\, \boldsymbol{\epsilon}_1 + (1-\lambda)\boldsymbol{\epsilon}_2]^T \mathbf{f}_0 < 0\ ,\\
&\lambda\, \boldsymbol{\epsilon}_1^T \mathbf{H}_1 \boldsymbol{\epsilon}_1 + (1-\lambda) \boldsymbol{\epsilon}_2^T \mathbf{H}_1 \boldsymbol{\epsilon}_2 + 2[\lambda\, \boldsymbol{\epsilon}_1 + (1-\lambda)\boldsymbol{\epsilon}_2]^T \mathbf{f}_1 > 0\ .
\end{aligned} \tag{13}$$

Let $\boldsymbol{\epsilon}_1 = \kappa\, \mathbf{z}$ and $\boldsymbol{\epsilon}_2 = \nu\, \mathbf{z}$, where κ and ν are infinitesimally small real numbers, and $\mathbf{z}$ is a K-dimensional real vector. Then, the conditions in (13) can be simplified, after some manipulation, as

$$\left(f_0^{(2)}(\mathbf{x},\mathbf{z}) + c\, f_0^{(1)}(\mathbf{x},\mathbf{z}) \right) \Big|_{\mathbf{x}=\mathbf{0}} < 0\ , \tag{14}$$

$$\left(f_1^{(2)}(\mathbf{x},\mathbf{z}) + c\, f_1^{(1)}(\mathbf{x},\mathbf{z}) \right) \Big|_{\mathbf{x}=\mathbf{0}} > 0\ , \tag{15}$$

$$c \doteq \frac{2\,[\lambda\, \kappa + (1-\lambda)\, \nu]}{\lambda\, \kappa^2 + (1-\lambda)\, \nu^2}\ . \tag{16}$$

Since $f_j^{(1)}(\mathbf{x},\mathbf{z}) > 0$ at $\mathbf{x} = \mathbf{0}$ for $j = 0, 1$, (14) and (15) can also be expressed as

$$\left(f_0^{(2)}(\mathbf{x},\mathbf{z}) f_1^{(1)}(\mathbf{x},\mathbf{z}) + c f_0^{(1)}(\mathbf{x},\mathbf{z}) f_1^{(1)}(\mathbf{x},\mathbf{z}) \right) \Big|_{\mathbf{x}=\mathbf{0}} < 0, \tag{17}$$

$$\left(f_1^{(2)}(\mathbf{x},\mathbf{z}) f_0^{(1)}(\mathbf{x},\mathbf{z}) + c f_0^{(1)}(\mathbf{x},\mathbf{z}) f_1^{(1)}(\mathbf{x},\mathbf{z}) \right) \Big|_{\mathbf{x}=\mathbf{0}} > 0. \tag{18}$$

It is noted from (16) that c can take any real value by selecting appropriate $\lambda \in [0, 1]$ and infinitesimally small κ and ν values. Therefore, under the condition in (12), which states that the first term in (17) is smaller than the first term in (18), there always exists c that satisfies the improvability conditions in (17) and (18). □

Note that Proposition 3 employs only the first and second derivatives of F_0 and F_1 without requiring the calculation of $\mathbf{n}_1$ and $\mathbf{n}_2$ as in Proposition 2. In [1], an improvability condition is obtained for scalar observations (i.e., for $K = 1$) based only on $\frac{\partial F_j(x)}{\partial x}$ and $\frac{\partial^2 F_j(x)}{\partial x^2}$ terms for $j = 0, 1$. Hence, Proposition 3 extends the improvability result in [1] not only to the case of vector observations but also to a more generic condition that involves partial derivatives, $\frac{\partial^2 F_j(x)}{\partial x_l x_i}$, as well.

Another improvability condition that depends directly on F_0 and F_1 is provided in the following proposition.

Proposition 4: *The detector is improvable if $F_1(\mathbf{x})$ and $-F_0(\mathbf{x})$ are strictly convex at $\mathbf{x} = \mathbf{0}$.*

Proof: Consider the improvability conditions in (13). Let $\boldsymbol{\epsilon}_1 = -\boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}$ and $\lambda = 0.5$. Then, (13) becomes

$$\boldsymbol{\epsilon}^T \mathbf{H}_0 \boldsymbol{\epsilon} < 0\ , \quad \boldsymbol{\epsilon}^T \mathbf{H}_1 \boldsymbol{\epsilon} > 0\ . \tag{19}$$

Since $F_1(\mathbf{x})$ is strictly convex and $F_0(\mathbf{x})$ is strictly concave at $\mathbf{x} = \mathbf{0}$, $\mathbf{H}_1$ is positive definite and $\mathbf{H}_0$ is negative definite. Hence, there exists $\boldsymbol{\epsilon}$ that guarantees improvability. □

Finally, an improvability condition that depends on the first-order partial derivatives of $F_0(\mathbf{x})$ and $F_1(\mathbf{x})$ is derived in the following proposition, which can be considered as an extension of the improvability condition in [1].

Proposition 5: *Assume that $F_0(\mathbf{x})$ and $F_1(\mathbf{x})$ are continuously differentiable around $\mathbf{x} = \mathbf{0}$. The detector is improvable if there exists a K-dimensional vector $\mathbf{s}$ such that $\left(\sum_{i=1}^K s_i \frac{\partial F_1(\mathbf{x})}{\partial x_i}\right)\left(\sum_{i=1}^K s_i \frac{\partial F_0(\mathbf{x})}{\partial x_i}\right) < 0$ is satisfied at $\mathbf{x} = \mathbf{0}$, where s_i represents the ith component of $\mathbf{s}$.*

Proof: Consider the improvability conditions in (13). Let $\boldsymbol{\epsilon}_1 = \varsigma\, \mathbf{s}_1$ and $\boldsymbol{\epsilon}_2 = \varsigma\, \mathbf{s}_2$ where $\mathbf{s}_1$ and $\mathbf{s}_2$ are any K-dimensional real vectors and ς is an infinitesimally small positive real number. Then, it can be shown that when

$$[\lambda\, \mathbf{s}_1 + (1-\lambda)\, \mathbf{s}_2]^T \mathbf{f}_0 < 0 \quad \text{and} \quad [\lambda\, \mathbf{s}_1 + (1-\lambda)\, \mathbf{s}_2]^T \mathbf{f}_1 > 0 \tag{20}$$

are satisfied, one can find an infinitesimally small positive ς such that the conditions in (13) are satisfied. Let $\mathbf{s} \doteq \lambda\, \mathbf{s}_1 + (1-\lambda)\, \mathbf{s}_2$. Note that $\mathbf{s}$ can be any K-dimensional real vector for suitable values of $\mathbf{s}_1$, $\mathbf{s}_2$ and $\lambda \in [0, 1]$. Based on the definition of $\mathbf{s}$, (20) can be expressed as $\mathbf{s}^T \mathbf{f}_0 < 0$ and $\mathbf{s}^T \mathbf{f}_1 > 0$.

For $\varsigma < 0$, similar argument can be used to show that $\mathbf{s}^T \mathbf{f}_0 > 0$ and $\mathbf{s}^T \mathbf{f}_1 < 0$ are sufficient conditions for improvability. Hence, $(\mathbf{s}^T \mathbf{f}_1)(\mathbf{s}^T \mathbf{f}_0) < 0$ can be obtained as the overall improvability condition. □

Comparison of the improvability conditions in this section with those in [8] reveals that the results in this section depend on functions F_0 and F_1 in (5) directly, whereas those in [8] are obtained based on $J(t)$ defined in (6). Therefore, this study provides a direct way of evaluating the improvability of a detector. However, the approach in [8] can be more advantageous in certain cases, as it deals with a single-variable function irrespective of the dimension of the data vector.

One application of the improvability results studied in this section is related to detection of communications signals in the presence of co-channel interference, which can result in Gaussian mixture noise at the receiver [12]. An example with Gaussian mixture noise is provided in the next section.

V. Numerical Results

In this section, a binary hypothesis-testing problem is studied to provide an example of the results presented in the previous sections. The hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ are defined as

$$\mathcal{H}_0\ :\ \mathbf{x} = \mathbf{w}\ , \quad \mathcal{H}_1\ :\ \mathbf{x} = A\mathbf{1} + \mathbf{w}\ , \tag{21}$$

where $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{1}$ denotes a vector of ones, $A > 0$ is a known scalar value, and $\mathbf{w}$ is Gaussian mixture noise with the following PDF

$$p_{\mathbf{W}}(\mathbf{x}) = \frac{1}{4\pi} \left[\frac{\mathrm{e}^{-\frac{1}{2}(\mathbf{x}+\boldsymbol{\mu})^T \boldsymbol{\Sigma}_1^{-1}(\mathbf{x}+\boldsymbol{\mu})}}{|\boldsymbol{\Sigma}_1|^{0.5}} + \frac{\mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}_2^{-1}(\mathbf{x}-\boldsymbol{\mu})}}{|\boldsymbol{\Sigma}_2|^{0.5}} \right], \tag{22}$$

where $\boldsymbol{\Sigma}_1 = \begin{bmatrix} \sigma^2 & \rho_1\sigma^2 \\ \rho_1\sigma^2 & \sigma^2 \end{bmatrix}$, $\boldsymbol{\Sigma}_2 = \begin{bmatrix} \sigma^2 & \rho_2\sigma^2 \\ \rho_2\sigma^2 & \sigma^2 \end{bmatrix}$, $\mathbf{x} = [x_1\ \ x_2]^T$, and $\boldsymbol{\mu} = [\mu_1\ \ \mu_2]^T$. In addition, the detector is described by

$$\phi(\mathbf{y}) = \begin{cases} 1\ , & y_1 + y_2 \geq A/2 \\ 0\ , & y_1 + y_2 < A/2 \end{cases}\ , \tag{23}$$

where $\mathbf{y} = \mathbf{x} + \mathbf{n}$, with $\mathbf{n}$ representing the additional independent noise term.

Based on (22), $F_0(\mathbf{x})$ and $F_1(\mathbf{x})$ can be calculated as follows:

$$F_i(\mathbf{x}) = \frac{1}{2}\, Q\left(\frac{A/2 - \gamma_2 - s_i}{\sigma\sqrt{2(1+\rho_1)}} \right) + \frac{1}{2}\, Q\left(\frac{A/2 - \gamma_1 - s_i}{\sigma\sqrt{2(1+\rho_2)}} \right), \tag{24}$$

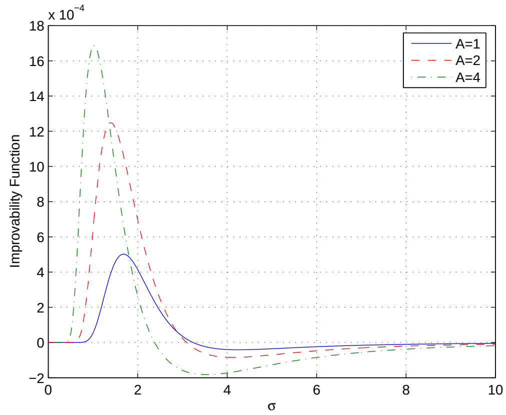

Fig. 1. The improvability function obtained from Proposition 3 for various values of A, where $\rho_1 = 0.1$, $\rho_2 = 0.2$, $\mu_1 = 2$, and $\mu_2 = 3$.

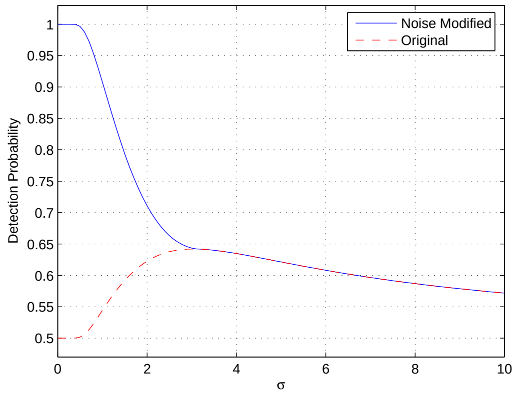

Fig. 2. Detection probabilities of the original and noise modified detectors versus σ for $A = 2$, $\rho_1 = 0.1$, $\rho_2 = 0.2$, $\mu_1 = 2$, and $\mu_2 = 3$.

for $i = 0, 1$, where $s_0 = 0$, $s_1 = 2A$, $\gamma_1 \doteq x_1 + x_2 + \mu_1 + \mu_2$, $\gamma_2 \doteq x_1 + x_2 - \mu_1 - \mu_2$, and $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \mathrm{e}^{-t^2/2} dt$ denotes the Q-function. From (24), the first and second derivatives can be obtained as

$$\frac{\partial F_i(\mathbf{x})}{\partial x_1} = \frac{\partial F_i(\mathbf{x})}{\partial x_2} = \frac{\mathrm{e}^{-\frac{(A/2-\gamma_2-s_i)^2}{4\sigma^2(1+\rho_1)}}}{4\sqrt{\pi}\sigma\sqrt{1+\rho_1}} + \frac{\mathrm{e}^{-\frac{(A/2-\gamma_1-s_i)^2}{4\sigma^2(1+\rho_2)}}}{4\sqrt{\pi}\sigma\sqrt{1+\rho_2}}$$

$$\frac{\partial^2 F_i(\mathbf{x})}{\partial x_1^2} = \frac{\partial^2 F_i(\mathbf{x})}{\partial x_2^2} = \frac{\partial^2 F_i(\mathbf{x})}{\partial x_1 \partial x_2} = \frac{\sigma^{-3}}{8\sqrt{\pi}} \left(\frac{(A/2-\gamma_2-s_i)}{\sqrt{(1+\rho_1)^3}} \times \mathrm{e}^{-\frac{(A/2-\gamma_2-s_i)^2}{4\sigma^2(1+\rho_1)}} + \frac{(A/2-\gamma_1-s_i)}{\sqrt{(1+\rho_2)^3}} \mathrm{e}^{-\frac{(A/2-\gamma_1-s_i)^2}{4\sigma^2(1+\rho_2)}} \right) \quad (25)$$

for $i = 0, 1$. It is noted from (25) that the first-order derivatives are always positive and all the first-order derivatives and the second-order derivatives are the same. Therefore, the improvability condition in (12) becomes independent of $\mathbf{z}$ for this example. Hence, the improvability condition in Proposition 3 can be stated as *when* $g(\sigma) \doteq \left[\frac{\partial^2 F_1(\mathbf{x})}{\partial x_1^2} \frac{\partial F_0(\mathbf{x})}{\partial x_1} - \frac{\partial^2 F_0(\mathbf{x})}{\partial x_1^2} \frac{\partial F_1(\mathbf{x})}{\partial x_1} \right] \Big|_{\mathbf{x}=\mathbf{0}}$ *is positive, the detector is improvable.* Fig. 1 plots the *improvability function* $g(\sigma)$ for various values of A. It is observed that the detector performance can be improved for $A = 1$ if $\sigma \in [0.55, 3.24]$, for $A = 2$ if $\sigma \in [0.42, 3.09]$, for $A = 4$ if $\sigma \in [0.29, 2.38]$. On the other hand, when the more generic result in Proposition 2 is applied to the same example, it is obtained that the detector is improvable for $A = 1$ if $\sigma \leq 3.24$, for $A = 2$ if $\sigma \leq 3.14$, and for $A = 4$ if $\sigma \leq 2.59$. Hence, Proposition 2 provides more generic improvability conditions as expected.

Fig. 2 plots the detection probabilities of the original (no additional noise) and the noise modified detectors with respect to σ for $A = 2$. For the noise modified detector, the optimal additional noise is calculated for each σ. For example, for $\sigma = 2$, the optimal additional noise is $p_{\mathbf{N}}(\mathbf{x}) = 0.6838\delta(\mathbf{x} - \mathbf{n}_1) + 0.3162\delta(\mathbf{x} - \mathbf{n}_2)$, where $\mathbf{n}_1 = [5.6668 \ \ -1.8180]$ and $\mathbf{n}_2 = [-1.3352 \ -4.6316]$. From the figure, it is observed that for smaller values of σ, more improvement is obtained, and after $\sigma = 3.14$ there is no improvement as expected from the improvability conditions.

In this specific example, it can be shown that the improvability conditions in Proposition 3 and in [8] are equivalent. Since the functions F_0 and F_1 defined in (24) are both monotone increasing functions of $x_1 + x_2$, $J(t) = \sup\{F_1(\mathbf{x}) \mid F_0(\mathbf{x}) = t\}$ can be obtained as $J(t) = \tilde{F}_1\left(\tilde{F}_0^{-1}(t)\right)$, where $\tilde{F}_i(m) \doteq F_i(\mathbf{x})\big|_{x_1+x_2=m}$. Then, $J^{''}(t) = \frac{d}{dt}\left\{ \frac{\tilde{F}_1^{'}(\tilde{F}_0^{-1}(t))}{\tilde{F}_0^{'}(\tilde{F}_0^{-1}(t))} \right\}$ can be obtained as

$$J^{''}(t) = \frac{\tilde{F}_1^{''}\left(\tilde{F}_0^{-1}(t)\right) - \frac{\tilde{F}_1^{'}(\tilde{F}_0^{-1}(t))\tilde{F}_0^{''}(\tilde{F}_0^{-1}(t))}{\tilde{F}_0^{'}(\tilde{F}_0^{-1}(t))}}{\left[\tilde{F}_0^{'}\left(\tilde{F}_0^{-1}(t)\right)\right]^2} . \quad (26)$$

At $t = \mathrm{P}_\mathrm{F}^\mathbf{x} = F_0(\mathbf{0}) = \tilde{F}_0(0)$, $\tilde{F}_0^{-1}(t)$ becomes equal to 0; hence, $J^{''}(\mathrm{P}_\mathrm{F}^\mathbf{x}) > 0$ implies $\tilde{F}_1^{''}(0) - \tilde{F}_0^{''}(0)\tilde{F}_1^{'}(0)/\tilde{F}_0^{'}(0) > 0$. For this specific problem, it can be shown that $\frac{d\tilde{F}_i(m)}{dm}\big|_{m=0} = \frac{\partial F_i(\mathbf{x})}{\partial x_1}\big|_{\mathbf{x}=\mathbf{0}} = \frac{\partial F_i(\mathbf{x})}{\partial x_2}\big|_{\mathbf{x}=\mathbf{0}}$ and $\frac{d^2\tilde{F}_i(m)}{dm^2}\big|_{m=0} = \frac{\partial^2 F_i(\mathbf{x})}{\partial x_1^2}\big|_{\mathbf{x}=\mathbf{0}} = \frac{\partial^2 F_i(\mathbf{x})}{\partial x_2^2}\big|_{\mathbf{x}=\mathbf{0}} = \frac{\partial^2 F_i(\mathbf{x})}{\partial x_1 \partial x_2}\big|_{\mathbf{x}=\mathbf{0}}$ for $i = 0, 1$, and $\frac{d\tilde{F}_0(m)}{dm}\big|_{m=0}$ is a positive constant. Therefore, the improvability conditions in Proposition 3 and that in [8] are equivalent in this specific example. However, it should be noted that the two conditions are not equivalent in general and the calculation of $J(t)$ can be difficult in the absence of monotonicity properties of F_0.

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