Fractional correlation

David Mendlovic, Haldun M. Ozaktas, and Adolf W. Lohmann

Recently, optical interpretations of the fractional-Fourier-transform operator have been introduced. On the basis of this operator the fractional correlation operator is defined in two different ways that are both consistent with the definition of conventional correlation. Fractional correlation is not always a shift-invariant operation. This property leads to some new applications for fractional correlation as shift-variant image detection. A bulk-optics implementation of fractional correlation is suggested and demonstrated with computer simulations.

Key words: Fourier optics, optical information processing, fractional Fourier transforms, correlation.

1. Motivation

Correlation is a useful tool for pattern recognition comparison, or search. It is perhaps the most important special case of convolution. Correlation is easily implemented optically, for example, with the VanderLugt 4-*f* coherent configuration,[1] with its analogous incoherent system,[2] or with the joint transform correlator.[3] The conventional correlation is a shift-invariant operation; thus shifting of the input pattern provides a shifted correlation output plane. In other words, when an input object is viewed as a collection of point sources, each point source in the object generates the same point-spread function in the output image independent of the point-source location. The location of the point-spread function at the output plane corresponds to the location of the point source at the input plane. In many cases this property is necessary, but sometimes not. An example is when one wants to obtain a correlation peak only when a specific object appears at a certain location (such as recognition of a stamp that could appear on a certain area of the envelope). Another example is when one wants to base the recognition decision mainly on the central pixels and less on the outer pixels.

Several approaches for obtaining such space-variance detection have been suggested. One of them used holographic filters that were made by use of reference beams with different angles.[4] Another approach was based on the use of different phase-encoded reference beams.[5] Recently a space-variant Fresnel-transform correlator was suggested.[6] This correlator is closely related to a lensless intensity correlator.[7]

In the following we suggest the use of the fractional Fourier transform (FRT) for implementing shift-variant pattern recognition. The FRT was defined mathematically by Namias.[8] Some of his mathematical derivations were incomplete and were later improved by McBride and Kerr.[9] Recently we defined[10–12] the FRT operator based on physical (optical) considerations. We discovered that our definition was equivalent to that given in Refs. 8 and 9. In these papers[10–12] we also showed how to realize optically the two-dimensional FRT as well as various mathematical and physical properties. Very recently, an alternative definition of the fractional Fourier transform was suggested[13] and shown to be equivalent to both previous definitions.[14] In retrospect this later definition emphasizes one of the most important properties of the FRT: its elegant presentation at the Wigner-distribution plane.

In Ref. 10 a new direction for generalizing the conventional correlation operation is mentioned briefly. It is based on the fractional Fourier transform and is thus coined fractional correlation. Conventional in this context means the standard Fourier mathematics.[15] In the following we extend the fractional correlation and investigate its use for object detection. As we will see, there is more than one way to define the fractional correlation based on the conventional correlation. We use computer simulations to demonstrate some simple examples of the options of using the fractional correlation operator.

D. Mendlovic is with the Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel. H. M. Ozaktas is with the Department of Electrical Engineering, Bilkent University, Ankara 06533, Bilkent, Turkey. A. W. Lohmann is with the Angewandte Optik, Erlangen University, Erlangen 8520, Germany.

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2. Notations and Definitions

2.A. Conventional Fourier Transform

Functions f and g are a Fourier pair if

$$F(\nu) = \int_{-\infty}^{\infty} f(x)\exp(-i2\pi\nu x)\mathrm{d}x, \tag{1}$$

$$f(x) = \int_{-\infty}^{\infty} F(\nu)\exp(i2\pi x\nu)\mathrm{d}\nu. \tag{2}$$

In operator notation we may write

$$F(\nu) = \mathcal{F} f(x). \tag{3}$$

It is well known that $\mathcal{F}^2 f(x) = f(-x)$ and $\mathcal{F}^4 f(x) = f(x)$, where $\mathcal{F}_j^i$ means application of $\mathcal{F}$ j times in succession.

2.B. Fractional Fourier Transform

Reference 10 describes the original fractional-Fourier-transform definition, which is based on the Hermite–Gaussian functions (the self modes of a quadratic graded-index medium). For the following analysis the Wigner-distribution interpretation of the fractional Fourier transform is more convenient because its optical interpretation contains bulk-optics elements that provide a high space–bandwidth product for comparing the graded-index elements. In fact, it was proposed as alternative definition of the fractional Fourier transform[13] and later proved[14] to be equivalent to the Hermite–Gaussian function definition. This definition states that performing the Pth fractional-Fourier-transform operation corresponds to rotating the Wigner-distribution by an angle

$$\phi = P(\pi/2) \tag{4}$$

in the clockwise direction. Detailed discussion of the Wigner distribution may be found in Refs. 16–20. Figure 1 shows the suggested optical setup for performing a fractional Fourier transform of order P. It contains two lenses with the focal length $f = f_1/\tan(\phi/2)$ with a space of $z = f_1[\sin(\phi)]$ between them. f_1 is a free parameter. When $P = 1$ and $\phi = \pi/2$, $f = z = f_1$, which is related to the classical Fourier transform. The effect of propagation of a signal through this setup is equivalent to performing an FRT of order P ($\mathcal{F}^P$) and can be expressed as

$$\mathcal{F}^P[u_0(x)] = u_P(x) = \int_{-\infty}^{\infty} u_o(x_0)\exp\left[i\pi\left(\frac{x^2 + x_0^2}{T}\right)\right] \times \exp\left(-i2\pi\frac{xx_0}{S}\right)\mathrm{d}x_0, \tag{5}$$

with

$$T = \lambda f_1 \tan(\phi), \quad S = \lambda f_1 \sin(\phi), \tag{6}$$

where λ is the wavelength of the incident light.

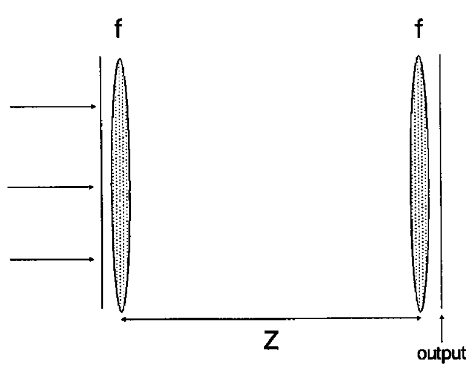

Fig. 1. Bulk-optics setup for performing a fractional Fourier transform of order P. f and z depend on P.

2.C. Conventional Correlation

The conventional correlation of $u_0(x)$ and $v_0(x)$ is defined as

$$C_1(x) = \int_{-\infty}^{\infty} u_0(x_0)v_0^*(x - x)\mathrm{d}x_0 = \int_{-\infty}^{\infty} u_1(\nu)v_1^*(\nu)\exp(i2\pi\nu x)\mathrm{d}\nu, \tag{7}$$

where

$$u_1(\nu) = \mathcal{F}^1 u_0(x), \quad v_1(\nu) = \mathcal{F}^1 v_0(x). \tag{8}$$

For what follows, the spectral definition of the correlation [see Eq. (7)] includes the following: Perform the first Fourier transform of both objects, take the complex conjugate of one of the objects, multiply the results, and finally, perform an inverse Fourier transform.

3. Basic Properties Proposed for the Definition of Fractional Correlation

Three basic requirements from the fractional correlation $C_P(x)$ are considered. The first one is mandatory:

$$\text{if } P = 1, \quad C_P(x) \rightarrow C_1(x). \tag{9}$$

In addition, we consider two weaker requirements whose satisfaction is not as critical as postulate (9). One is connected with the autocorrelation center value for every P:

$$\text{if } v = u, \quad C_P(0) = C_1(0) = \int_{-\infty}^{\infty} |u_0(x_0)|^2\mathrm{d}x_0. \tag{10}$$

Note that u, v should be located at the same location, and thus the conventional correlation $C_1(x)$ obtains its maximum at $x = 0$ (while $v = u$). This means that $C_1(0) \geq C_1(x)$ for every x.

The third postulate ensures that $P = 0$ means a regular multiplication of u and v^*:

$$\text{if } P = 0, \quad C_0(x) = u_0(x)v_0^*(x). \tag{11}$$

In Section 4, two fractional correlation definitions that follow postulate (9) are presented.

4. Various Fractional Correlation Definitions

Before defining the fractional correlation operation, we first show what the steps are for performing the conventional correlation. Two approaches can be used for obtaining the conventional correlation. The first approach is as follows:

(1) Start with $u_0(x_0)$ and $v_0(x_0)$.
(2) Perform F^1 on both functions to obtain $u_1(y)$ and $v_1(y)$.
(3) Perform the complex conjugate of $v_1(y)$.
(4) Perform the multiplication $u_1(y)\ v_1^*(y)$ to obtain F^1C_1.
(5) Perform F^{-1} to obtain $C_1(x)$.

The second approach is as follows:

(1) Start with $u_0(x_0)$ and $v_0(x_0)$.
(2) Perform $v_0(x_0) \rightarrow v_0^*(-x_0)$.
(3) Perform F^1 on both functions to obtain $u_1(y)$ and $v_1^*(y)$.
(4) Multiply $u_1(y)v_1^*(y)$ to obtain $F^1C_1(x)$.
(5) Perform F^{-1} to obtain $C_1(x)$.

It is a fact that both approaches lead to the same output $C_1(x)$. In both processes, by replacing the F^1 and F^{-1} operators with those of fractional order F^{P1} and F^{P2}, respectively, we can define the fractional correlation operator in such a way as to fulfill mandatory postulate (9). However, the two definitions are not necessarily identical for $P_1, P_2 \neq 1$. For example, to check postulate (11), we should replace all the Fourier-transform operations with F^0, which is the identity operator. Thus the first approach results in $C_0(x) = u_0(x)v_0^*(x)$ and the second in $C_0(x) = u_0(x)v_0^*(-x)$. The second result is different from postulate (11). Of course, postulate (11) can be modified to fit the second approach.

Now let us look at an interesting property of the FRT:

$$F^P v_0^*(-x) = v_{-P}^*(-x) = v_{2-P}^*(x). \tag{12}$$

With this relation, steps 2 and 3 of the second approach can be replaced by the following step: Perform $F^P u_0(x)$ and $[F^{2-P}v_0(x)]^*$.

5. Generic Form of the Fractional Correlation Output

Now let us write explicitly the output signal $C_P(x)$. In order to investigate the most general case, we assume $P = P_1$ for the fractional Fourier operators before the multiplications (step 4 in both approaches) and $P = P_2$ for the fractional Fourier transform after the multiplications. It is not necessary that $P_1 = P_2$, and thus we denote the correlation output C_{P_1,P_2}. When Eq. (5) is substituted instead of the F operator, C_{P_1,P_2} becomes

$$C_{P_1,P_2}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x_0)v_0^*(\pm\tilde{x}_0) \times \exp[i\pi\Psi_1(x, x_0, \tilde{x}_0, y)] \times \exp[-i2\pi\Psi_2(x, x_0, \tilde{x}_0, y)]\mathrm{d}x_0\mathrm{d}\tilde{x}_0\mathrm{d}y, \tag{13}$$

with

$$\Psi_1(x, x_0, \tilde{x}_0, y) = \frac{x^2 + y^2}{T_2} + \frac{x_0^2 + y^2}{T_1} \mp \frac{\tilde{x}_0^2 + y^2}{T_1}, \tag{14}$$

$$\Psi_2(x, x_0, \tilde{x}_0, y) = y\left(\frac{x_0 \mp \tilde{x}_0}{S_1} + \frac{x}{S_2}\right), \tag{15}$$

$$T_1 = \lambda f_1 \tan(\phi_1), \quad T_2 = \lambda f_1 \tan(\phi_2), \tag{16}$$

$$S_1 = \lambda f_1 \sin(\phi_1), \quad S_2 = \lambda f_1 \sin(\phi_2), \tag{17}$$

$$\phi_1 = P_1(\pi/2), \quad \phi_2 = P_2(\pi/2), \tag{18}$$

while f_1 is a constant. Regarding the $\pm$ and $\mp$ symbols, the upper symbol is for the first approach, while the lower symbol is for the second approach.

6. Special Cases

6.A. Symmetric Case

We now want to reduce the triple integral of Eq. (13). The variable y is the only one that does not occur in the object functions u_0 and v_0^* of the integrand. Hence, by using a well-known finite integral,[21] the saddle-point integration method $\int \ldots \mathrm{d}y$ can be estimated, and Eq. (13) becomes

$$C_{P_1,P_2}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x_0)v_0^*(\tilde{x}_0)\frac{\exp(-i\pi/4)}{\left(\left|\frac{1}{T_2} + \frac{1 \mp 1}{T_1}\right|\right)^{1/2}} \times \exp\left\{i\pi\left[\left(\frac{x^2}{T_2} + \frac{x_0^2 \mp \tilde{x}_0^2}{T_1}\right) + \frac{\left(\frac{x_0 \mp \tilde{x}_0}{S_1} + \frac{x}{S_2}\right)^2}{\frac{1}{T_2} + \frac{1 \mp 1}{T_1}}\right]\right\} \times \mathrm{d}x_0\mathrm{d}\tilde{x}_0. \tag{19}$$

Unfortunately, it is complicated to reduce the last general expression of C_{P_1,P_2} to a single integral form as in the conventional correlation expression [see Eq. (7)]. However, in Subsections 6.B and 6.C we present two special cases in which the final fractional correlation expression is a single integral.

Let us consider the symmetric case of $P_2 = -P_1$. Here

$$T_2 = -T_1, \quad S_2 = -S_1, \tag{20}$$

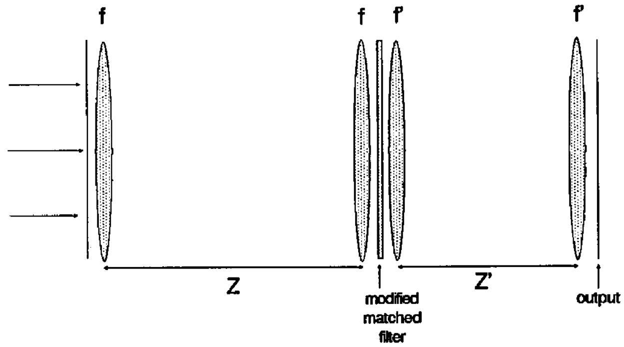

Fig. 2. Two optical fractional Fourier transformers in cascade for performing the fractional correlation. The first fractional Fourier transformer is for order P_1, and the second is for P_2. The output is $C_{P_1,P_2}(x)$.

and thus for this case Eq. (19) is

$$C_{P_1,P_2}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x_0)v_0^*(\tilde{x}_0)\,\frac{\exp(-i\pi/4)}{(|1/T_1|)^{1/2}} \times \exp\left\{i\pi\left[\left(\frac{-x^2 + x_0^2 \mp \tilde{x}_0^2}{T_1}\right) \mp T_1\left(\frac{x_0 \mp \tilde{x}_0 - x}{S_1}\right)^2\right]\right\}\mathrm{d}x_0\mathrm{d}\tilde{x}_0. \tag{21}$$

Equation (21) is shorter than Eq. (19) but still contains two integrals. The above symmetric case should be considered as the most logical way for defining the fractional correlation operator because of its similarity to the conventional correlation. However, the nonsymmetrical definitions that are discussed in Subsections 6.B and 6.C lead to single integral expressions, which is certainly desirable.

6.B. Modified Case I

Let us now investigate the case in which $\int \ldots \mathrm{d}y$ is a Dirac integral. We can achieve this by choosing T_1 and T_2 such that no y^2 occurs in the exponent. Application of this condition to Eq. (14) when the first definition of fractional correlation is used yields

$$1/T_2 = 0, \tag{22}$$

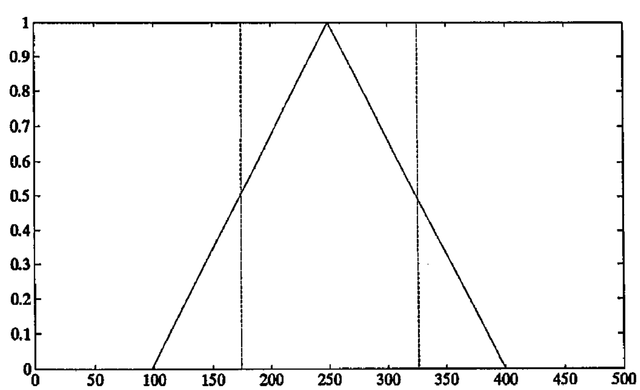

Fig. 3. Input signal (dashed curve) and its conventional autocorrelation signal (solid curve) C_1.

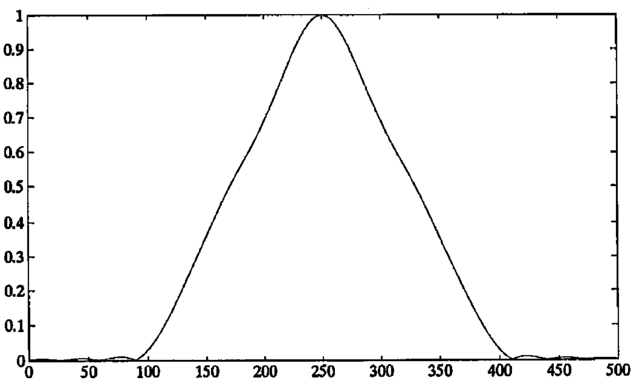

Fig. 4. Fractional autocorrelation of order $P_1 = 0.9$ according to the first definition special case.

or

$$P_2 = 1, \quad S_2 = \lambda f. \tag{23}$$

On the basis of

$$\int_{-\infty}^{\infty} \exp\left[-i2\pi y\left(\frac{x_0 - \tilde{x}_0}{S_1} + \frac{x}{S_2}\right)\right]\mathrm{d}y = \delta\left(x_0 - \tilde{x}_0 + \frac{S_1}{S_2}x\right), \tag{24}$$

one obtains

$$C_{P_1,1}(x) = \exp\left[-i\pi\,\frac{(S_1/S_2)^2}{T_1}\,x^2\right]\int_{-\infty}^{\infty} u_0(x_0)v_0^* \times \left(x_0 + \frac{S_1}{S_2}x\right)\exp\left(-\frac{i2\pi}{T_1}\frac{S_1}{S_2}xx_0\right)\mathrm{d}x_0 \tag{25}$$

as the final formula for fractional correlation. In this case we have $S_1/S_2 = \sin(\phi_1)$ and $S_1/(T_1S_2) = \cos[\phi_1/(\lambda f)]$.

Another nice feature of this modification is that postulate (10) is fulfilled because

$$C_{P_1,1}(0) = \int_{-\infty}^{\infty} u_0(x_0)u_0^*(x_0)\mathrm{d}x_0 = \int_{-\infty}^{\infty} |u_0(x_0)|^2\mathrm{d}x_0. \tag{26}$$

For this case, postulate (11) is not fulfilled.

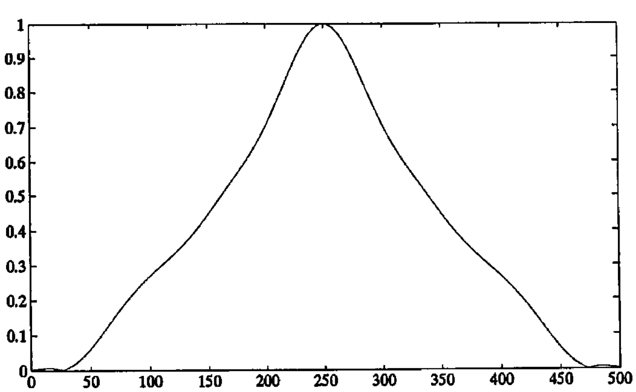

Fig. 5. Same as Fig. 4 but for $P_1 = 0.5$.

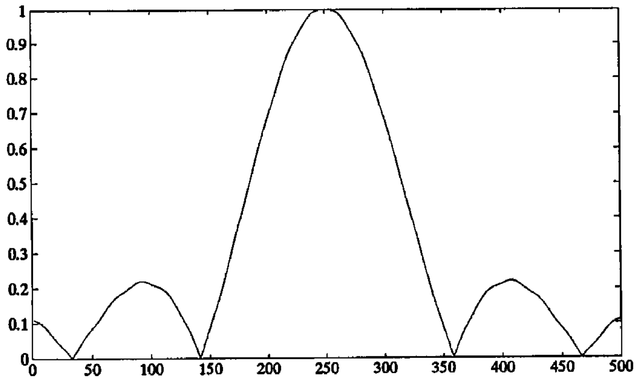

Fig. 6. Same as Fig. 4 but for $P_1 = 0$.

6.C. Modified Case II

When the second definition of fractional correlation is applied, the y integral of Eq. (13) is a Dirac integral if

$$T_2 = -T_1/2, \tag{27}$$

and we obtain

$$\int_{-\infty}^{\infty} \exp\left[-i2\pi y\left(\frac{x_0 + \tilde{x}_0}{S_1} + \frac{x}{S_2}\right)\right]\mathrm{d}y = \delta\left(x_0 + \tilde{x}_0 + \frac{S_1}{S_2}x\right)\cdot \tag{28}$$

The output is

$$C_{P_1,P_2}(x) = \exp\left[-i\pi\,\frac{(S_1/S_2)^2}{T_1}\,x^2\right]\int_{-\infty}^{\infty} u_0(x_0)v_0^*\left(-x_0 - \frac{S_1}{S_2}x\right)\exp\left(-\frac{i2\pi}{T_1}\frac{S_1}{S_2}xx_0\right)\mathrm{d}x_0. \tag{29}$$

Equation (29) has a form similar to Eq. (25). Although postulates (10) and (11) are not fulfilled now, they can be modified for this special case. Postulate (10) may now be spelled out as the following:

$$\text{if } u = v, \quad C_{P_1,P_2}(0) = C_1(\mathrm{x}) = \int_{-\infty}^{\infty} u_0(x_0)u_0^*(-x_0)\mathrm{d}x. \tag{30}$$

Postulate (11) may now be written as the following:

$$\text{if } P_1 = P_2 = 0, \quad C_{0,0}(x) = u_0(x)v_0^*(-\mathrm{x}). \tag{31}$$

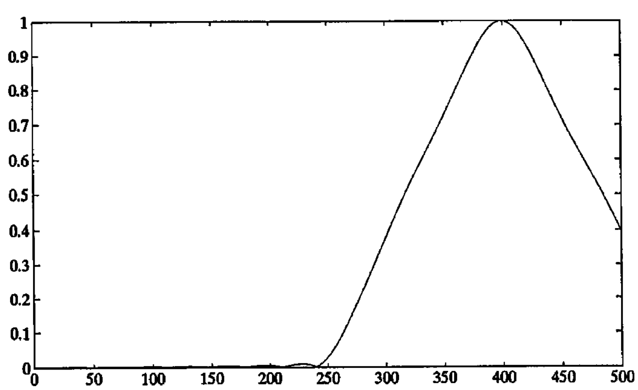

Fig. 7. Fractional correlation of order $P_1 = 0.9$ according to the first definition special case with a shift of 150 pixels of the input signal.

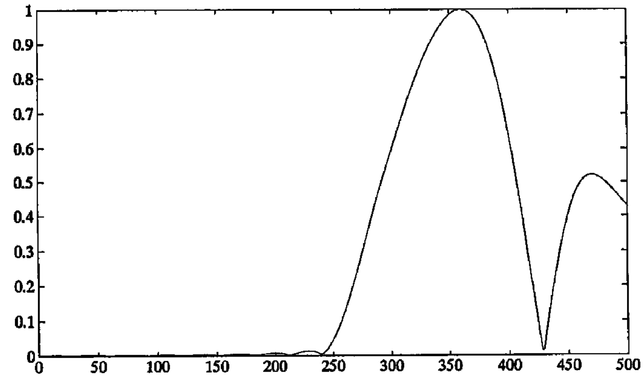

Fig. 8. Same as Fig. 7 but with $P_1 = 0.5$.

7. Optical Implementation

Figure 1 shows the optical setup for performing a fractional Fourier transform of order P. Following the definition of the fractional correlation, one can generate a modified matched filter

$$H_1(\mathrm{x}) = \{F^{P_1}\mathrm{v}(\mathrm{x})\}^* \tag{32}$$

for the first approach and

$$H_2(x) = F^{P_1}v^*(-x) \tag{33}$$

for the second approach. This modified matched filter is placed between two optical fractional Fourier transformers, as shown in Fig. 2. The output is C_{P_1,P_2}.

How can we generate the modified matched filter? There are two possibilities:

(1) We can use computer-generated hologram tech-

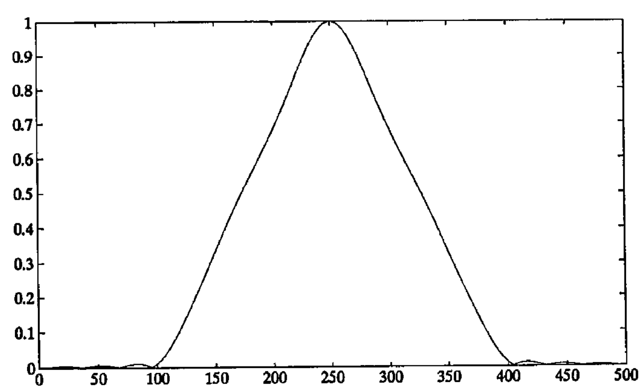

Fig. 9. Fractional autocorrelation of order $P_1 = 0.9$ according to the second definition special case.

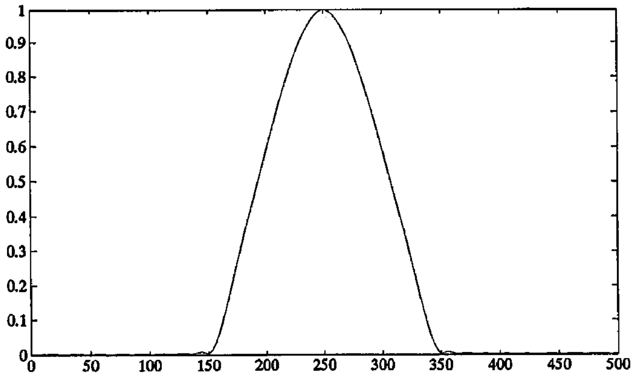

Fig. 10. Same as Fig. 9 but with $P_1 = 0.5$.

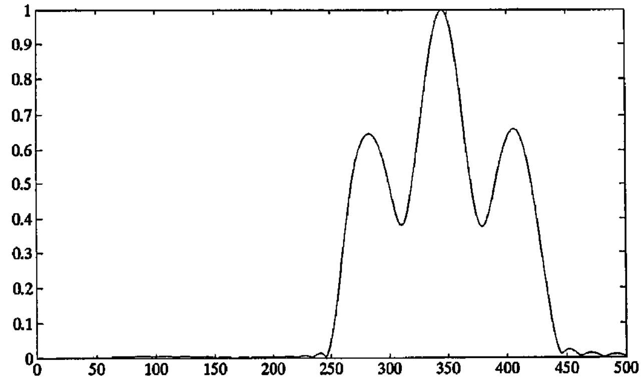

Fig. 12. Fractional correlation of order $P_1 = 0.5$ according to the second definition special case with a shift of 150 pixels of the input signal.

niques. Because in general the modified matched filter is a complex amplitude function, only phase and amplitude coding techniques are satisfactory, such as the detour phase method.[22]

(2) We can use direct holographic means in a way similar to that suggested by VanderLugt[1] for recording the conventional matched filter. The reference object is placed at the input of the setup of Fig. 1, and the output is illuminated with a tilted plane wave as a reference beam. At the output a holographic plate is placed that, after exposure, is the matched filter.

In both methods the fractional correlation signal C_{P_1,P_2} is obtained along the first diffraction order.

8. Computer Simulations

In order to illustrate the use of fractional correlation, we performed computer simulations according to the optical setup of Fig. 2 using a MATLAB subroutine. The FRT was computed based on the Hermite–Gaussian modes FRT definition; we did not follow the Wigner FRT definition. This was done for the sake of shorting the computing time. We simulated only the two special cases that were introduced in Section 6. Simulations of the fractional Fourier transform itself can be found in Ref. 11. First, we simulated the conventional correlation. Figure 3 shows the input signal (a rect function) and its conventional autocorrelation. With the first definition, exactly the same result is obtained for $P_1 = 1$. Now, let us reduce P_1. Figures 4, 5, and 6 are the fractional autocorrelation following the first definition with orders P_1 equal to 0.9, 0.5, and 0, respectively ($P_2 = 1$ for this case). Figures 7 and 8 are the 0.9 and 0.5 fractional correlation output after shifting of one of the signals with 150 pixels. By inspection one can notice that for the $P_1 = 0.5$ case the fractional correlation is not shift invariant.

Similar simulations were performed for the second definition. Figure 3 (the conventional correlation) is the fractional autocorrelation of order $P_1 = 1$ according to the second definition. Figures 9, 10, and 11 are the fractional autocorrelation of orders P_1 equal to 0.9, 0.5, and 0.2, respectively (here P_2 is calculated from P_1). Figures 12 and 13 are the correlations when one signal is shifted with 150 pixels for orders P_1 equal to 0.5 and 0.2. Again, the lack of the shift-invariance property is apparent. Figures 4–13 show clearly that the two fractional correlation definitions provide different results.

9. Conclusion

We have investigated the many various possibilities for defining the fractional correlation based on fractional-Fourier-transform operation. Two definitions were suggested, and each of them obtained a special case in which the fractional correlation mathematical expression is a single integral. An optical bulk-optics implementation was suggested that was

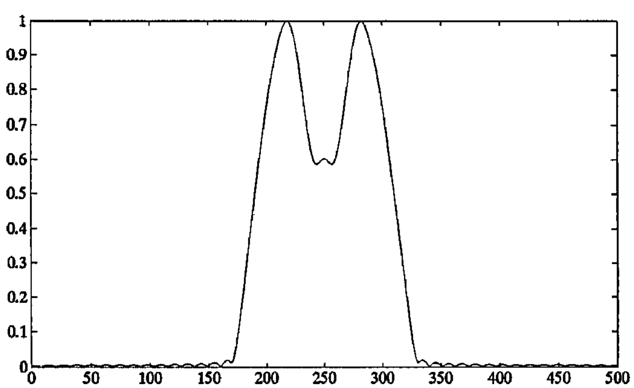

Fig. 11. Same as Fig. 9 but with $P_1 = 0.2$.

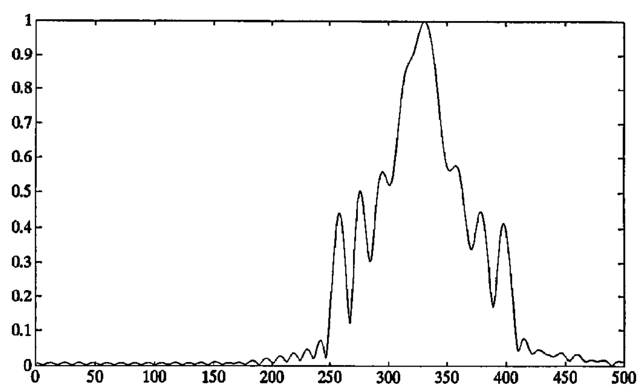

Fig. 13. Same as Fig. 12 but with $P_1 = 0.2$.

very similar to the conventional 4-f correlator. Computer simulations demonstrated that the fractional correlation operator is sometimes not a shift-invariant operator. In a similar way the fractional convolution can be defined as discussed in Ref. 23. In a future study the usefulness of this new operator for object detection will be checked according to important criteria such as signal-to-noise ratio, peak height, and light efficiency.

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