Space–bandwidth-efficient realizations of linear systems

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One can obtain either exact realizations or useful approximations of linear systems or matrix–vector products that arise in many different applications by implementing them in the form of multistage or multichannel fractional Fourier-domain filters, resulting in space–bandwidth-efficient systems with acceptable decreases in accuracy. Varying the number and the configuration of filters enables one to trade off between accuracy and efficiency in a flexible manner. The proposed scheme constitutes a systematic way of exploiting the regularity or structure of a given linear system or matrix, even when that structure is not readily apparent.

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Let $f(\mathbf{r})$ denote the input and $g(\mathbf{r}) = \int H(\mathbf{r}, \mathbf{r}')f(\mathbf{r}')\mathrm{d}\mathbf{r}'$ the output of an arbitrary linear system characterized by the kernel $H(\mathbf{r}, \mathbf{r}')$, where $\mathbf{r} = (x, y)$. The same system can be written in operator notation as $g = \mathcal{H}f$. In some applications, such as image enhancement, we may wish to implement a linear system that is deliberately designed to impart a certain effect to the input. In others, such as image restoration or reconstruction, linear systems are used to recover a desired image from whatever data or measurements are available. If we are working with images whose space–bandwidth products are $N = \sqrt{N} \times \sqrt{N}$, the above system can be approximated with the discrete form $g_{kl} = \sum_{j=1}^{\sqrt{N}} \sum_{i=1}^{\sqrt{N}} H_{klij}f_{ij}$. This expression can represent either a tensor or a matrix algebra operation that we wish to implement or an approximation of its continuous counterpart. Digital implementation of such general linear systems takes $O(N^2)$ time. Common single-stage optical implementations, such as optical matrix–vector multiplier or multifacet architectures,[1] require an optical system whose space–bandwidth product is $O(N^2)$.

The output of a space-invariant system characterized by the impulse response $h(\mathbf{r})$ is related to the input by the convolution relation $g(\mathbf{r}) = \int h(\mathbf{r} - \mathbf{r}')f(\mathbf{r}')\mathrm{d}\mathbf{r}'$, which can again be expressed in tensor or matrix notation, but this time the tensor or matrix is of a special form (Toeplitz). Digital implementation of such systems take $O(N \log N)$ time (by use of a fast Fourier transform). Optical implementation requires an optical system whose space–bandwidth product is $O(N)$.

Because of the intrinsic nature of some problems, convolution-type systems are fully adequate. However, in other cases, the use of such systems is either totally inappropriate or at best yields a crude approximation. We may think of space-invariant systems and general linear systems as representing two extremes in a cost–accuracy trade-off. Sometimes the use of space-invariant systems may be inadequate, but at the same time the use of general linear systems may be overkill and prohibitively costly. In this Letter we consider multistage and multichannel fractional Fourier-domain filtering configurations that interpolate between these two extremes, offering greater flexibility in trading off between cost and accuracy. Common single-stage frequency-domain multiplicative filtering is shown in Fig. 1(a). The dual of this operation is single-stage space-domain multiplicative filtering and is shown in Fig. 1(b). Figure 1(c) depicts single-stage multiplicative filtering in the ath-order fractional Fourier domain. This filtering configuration and its applications are discussed in Refs. 2–5.

The ath-order fractional Fourier transform[6,7] of $f(x)$ is denoted as $\mathcal{F}^a f(x)$. Then, $\mathcal{F}^0 f(x) = f(x)$ itself, $\mathcal{F}^1 f(x) = F(\nu)$, the ordinary Fourier transform, and $\mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_2+a_1}$. The ath fractional Fourier domain makes an angle $\alpha = a\pi/2$ with the space domain in the space–frequency plane [Fig. 1(d)].[6,8] Generalization to two dimensions is straightforward.[9]

In the multistage system [Fig. 1(e)] that was suggested in Ref. 6, the input is first transformed into the a_1th domain, where it is multiplied by a filter $h_1(\mathbf{r})$. The result is then transformed back into the original domain. This process is repeated M times. The multichannel filter structure consists of M single-stage blocks in parallel [Fig. 1(f)]. For each channel k, the input is transformed to the a_kth domain, multiplied by a filter $h_k(u)$, and then transformed back.

Here, we let Λ_j denote the operator corresponding to multiplication by the filter function $h_j(\mathbf{r})$. Then the outputs g_s and g_p of the serial and the parallel configurations, respectively, are related to the input f

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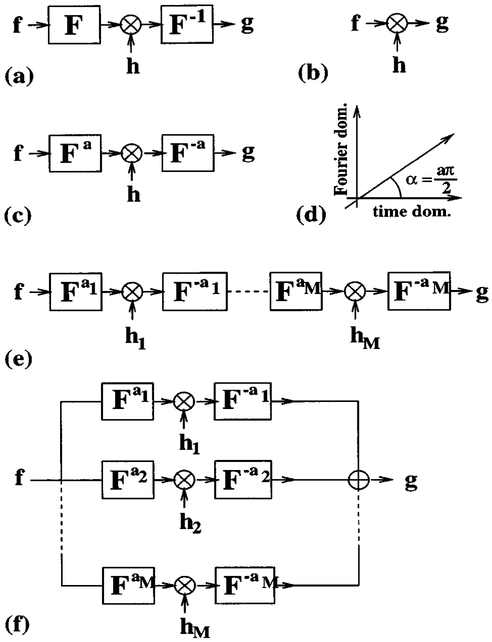

Fig. 1. (a) Fourier-domain filtering. (b) Space-domain filtering. (c) ath-order fractional Fourier-domain filtering. (d) ath-order fractional Fourier domain. (e) Multistage (serial) filtering. (f) Multichannel (parallel) filtering.

by

$$g_s = (\mathcal{F}^{-a_M}\Lambda_M \ldots \mathcal{F}^{a_2-a_1}\Lambda_1\mathcal{F}^{a_1})f\,, \tag{1}$$

$$g_p = \left(\sum_{k=1}^{M} \mathcal{F}^{-a_k}\Lambda_k\mathcal{F}^{a_k}\right)f\,, \tag{2}$$

where $\mathcal{F}^{a_j}$ represents the a_jth-order fractional Fourier transform operator. Equations (1) and (2) can also be expressed as $g = \mathcal{T}f$, where $\mathcal{T}$ is the operator representing the overall filtering configuration.

Both filtering configurations have at most $MN + M$ degrees of freedom. Their digital implementation will take $O(MN \log N)$ time, since the fractional Fourier transform can be implemented in $O(N \log N)$ time.[10] Optical implementation will require an M-stage or an M-channel optical system, each with space–bandwidth product N.[11] These configurations interpolate between general linear systems and shift-invariant systems in terms of both cost and flexibility. If we choose M to be small, cost and flexibility are both low. If we choose a larger M, cost and flexibility are both higher; as M approaches N, the number of degrees of freedom approaches that of a general linear system.

Increasing M allows us to approximate a given linear system better. For a given value of M, we can approximate this system with a certain degree of accuracy (or error). For instance, a shift-invariant system can be realized with perfect accuracy with $M = 1$. In general there will be a finite accuracy for each value of M. As M is increased, the accuracy will usually increase but never decrease, because systems with larger values of M include systems with smaller values of M as their special cases. In dealing with a specific application, we can seek the minimum value of M that results in the desired accuracy or the highest accuracy that can be achieved for a given M. Thus these systems give us considerable freedom in trading off efficiency and accuracy, enabling us to seek the best performance for a given cost or the least cost for a given level of performance.

Naturally, the value of M required for attaining a given accuracy will be smaller for system whose kernels exhibit greater regularity. In such cases direct implementation of the linear system is clearly inefficient. The inherent structure can be exploited on a case-by-case basis through ingenuity or invention; the most-efficient algorithms and optical transforms are obtained in this manner. In contrast, our method provides a systematic way of obtaining an efficient implementation that does not require ingenuity on a case-by-case basis. This approach would be especially useful when the regularity or the structure of the matrix is not simple or immediately apparent. A distinct circumstance in which the method can be beneficial is when it is sufficient to implement the linear system with limited accuracy, as may be the case when some other component of the overall system limits the accuracy to a lower value anyway, or simply when the application itself demands limited accuracy.

The scheme suggested here offers improvements in both digital and optical systems. We expect it to be especially beneficial in optical systems, since the cost of optical components is a strong function of their space–bandwidth product. Direct single-stage implementation of linear systems that use matrix–vector products or multifacet architectures may be totally unfeasible (a 256×256 image would require a space–bandwidth product of 256^4). Furthermore, the intrinsic amplitude- or intensity-level accuracy of analog optical systems is usually limited to quite-modest dynamic ranges. Given this limited accuracy, it is pointless to try to implement the desired linear system by use of an expensive scheme that could in principle accommodate much greater accuracies (such as a conventional matrix–vector multiplier architecture that maps an exact matrix–vector product). Our proposal allows one to approximate the desired linear system to an accuracy that just matches the intrinsic accuracy of analog optical systems in a manner that reduces the cost as much as possible. When we employ analog optical systems to image digital optical signals (e.g., free-space optical interconnections), even-lower accuracy is tolerable. Here it is possible to make do with even-smaller numbers of stages, since because of the digital nature of the signals even greater deviation of the obtained matrix $\mathbf{T}$ from the specified matrix $\mathbf{H}$ can be tolerated while an acceptable eye pattern is maintained. For these reasons, the proposed scheme would be especially useful in optical systems, including analog signal-processing systems, matrix processors, numerical processors, algebraic processors, and optical interconnections.

The proposed system can be used in one of two distinct ways.[12,13] Here we have space for only one of them: We take Eq. (1) or (2) as a constraint on the

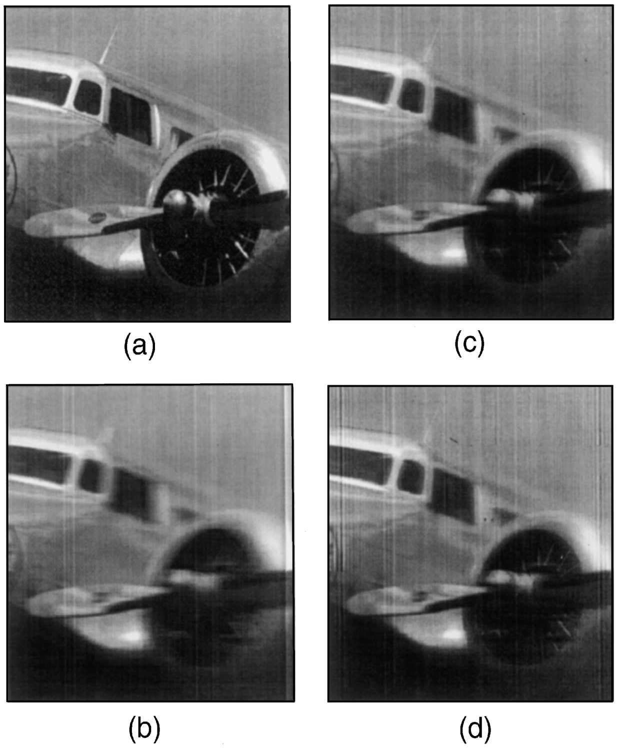

Fig. 2. (a) Original image. (b) Blurred image. (c) Image restored by multistage filtering. (d) Image restored by multichannel filtering.

form of the linear-estimation matrix. Given a specific optimization criterion, such as minimum mean-square error, we find the optimal values of a_j and $\mathbf{h}_j$ such that the given criterion is optimized.

The repeated filtering configuration is discussed at length in Refs. 12 and 13, in which many examples are given of instances in which it is possible to obtain useful approximations by use of only a moderate number of stages. Here we must satisfy ourselves with three examples.

Our first example[12] is the restoration of signals that were degraded by space-variant atmospheric turbulence by use of repeated filtering. The resulting mean-square error between the actual signal and its estimate is found to be 1% with $M = 4$.

In our second example we consider restoration of images blurred by a space-variant point-spread function (Fig. 2). The mean-square estimation error is 10% in the multistage case and 13% in the multichannel case for $M = 4$ and 3.7% in the multistage case and 5.5% in the multichannel case for $M = 8$. Ordinary Fourier-domain filtering gives poor results, resulting in an error of 34%.

In this example the orders were chosen to be equally spaced between 0 and 1. Optimizing over the orders will always yield better results but can be difficult when M is large. (How to choose a set of orders $a_1, a_2, \ldots, a_M$ that are suitable for given application classes or have certain desirable properties is the subject of current research.)

Finally, we consider the problem of recovering a signal consisting of multiple chirplike components buried in white Gaussian noise with a signal/noise ratio of 0.1. We assume that the signal consists of six chirps with uniformly distributed random amplitudes and time shifts and that the chirp rates are known with a ±5% accuracy. Use of $M = 6$ filters results in an estimation error of 2.6%.

We employed an iterative algorithm to determine the optimal filters $\mathbf{h}_j$ that minimize mean-square error. The repeated system is highly nonlinear in the filter coefficients and does not admit of an exact solution. The multichannel system is linear in the filter coefficients and has an exact solution, but in practice an iterative solution is preferred.

The multistage and multichannel configurations may be based on other transforms with efficient implementations, such as linear canonical transforms.[14] Concentrating on Eq. (2), the essential idea is to approximate a general linear operator as a linear combination of operators with fast algorithms or space–bandwidth-efficient implementations. If an acceptable approximation can be found with a value of M that is not too large, the cost can be significantly reduced.

The serial and parallel filtering configurations can be combined in an arbitrary manner to give what we term generalized filtering configurations or circuits. What types of circuit may be beneficial in what circumstances is an area for future research.

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