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The Fractional Fourier Transform and Harmonic Oscillation

M. ALPER KUTAY
The Scientific and Technical Research Council of Turkey, The National Research Institute of Electronics and Cryptology (TUBITAK – UEKAE), Atatürk Bulvarı no 221, 06100 Kavaklıdere, Ankara, Turkey

HALDUN M. OZAKTAS
Department of Electrical Engineering, Bilkent University, 06533 Bilkent, Ankara, Turkey

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Abstract. The ath-order fractional Fourier transform is a generalization of the ordinary Fourier transform such that the zeroth-order fractional Fourier transform operation is equal to the identity operation and the first-order fractional Fourier transform is equal to the ordinary Fourier transform. This paper discusses the relationship of the fractional Fourier transform to harmonic oscillation; both correspond to rotation in phase space. Various important properties of the transform are discussed along with examples of common transforms. Some of the applications of the transform are briefly reviewed.

Keywords: Fractional Fourier transform, harmonic oscillation, Green's function, phase space.

1. Introduction

The fractional Fourier transform is a generalization of the ordinary Fourier transform with an order parameter a such that the ath-order fractional Fourier transform operation corresponds to the ath power of the ordinary Fourier transform operation. Mathematically, if we denote the ordinary Fourier transform operator by $\mathcal{F}$, then the ath-order fractional Fourier transform operator is denoted by $\mathcal{F}^a$. A standard way of defining a function $G(\mathcal{A})$ of a linear operator $\mathcal{A}$ is to require that $G(\mathcal{A})$ have the same eigenfunctions $\phi_n(u)$ as $\mathcal{A}$, but eigenvalues which are the same function $G(\lambda_n)$ of the eigenvalues λ_n of $\mathcal{A}$. That is, if the eigenvalue equation of $\mathcal{A}$ is $\mathcal{A}\phi_n(u) = \lambda_n\phi_n(u)$, we define $G(\mathcal{A})$ through the eigenvalue equation $G(\mathcal{A})\phi_n(u) = G(\lambda_n)\phi_n(u)$. The eigenvalue equation of the ordinary Fourier transform operator is $\mathcal{F}\phi_n(u) = \exp(-in\pi/2)\phi_n(u)$, where $\phi_n(u)$, $n = 0, 1, 2, \ldots$ are the set of Hermite–Gaussian functions (Section 3). Now, particularizing the above general approach to the fractional power function, the fractional Fourier transform can be defined in terms of the following eigenvalue equation:

$$\mathcal{F}^a\phi_n(u) = [\exp(-in\pi/2)]^a\phi_n(u). \tag{1}$$

Since the fractional power function $[\cdot]^a$ is not uniquely defined, a choice must be made in evaluating $[\exp(-in\pi/2)]^a$. The choice leading to the definition which has so far received the most interest is

$$[\exp(-in\pi/2)]^a = \exp(-ian\pi/2).$$

Agreeing on this choice, we now show how to find the fractional Fourier transform of an arbitrary square-integrable function $f(u)$. First, we expand $f(u)$ in terms of the eigenfunctions of the Fourier transform:

$$f(u) = \sum_{n=0}^{\infty} C_n \phi_n(u) \quad \text{where} \quad C_n = \int \phi_n(u) f(u)\, \mathrm{d}u.$$

Since we know the action of $\mathcal{F}^a$ on the eigenfunctions $\phi_n(u)$, using linearity we can obtain the ath power fractional Fourier transform $f_a(u) \equiv \mathcal{F}^a f(u)$ of the original function $f(u)$ as

$$f_a(u) = \sum_{n=0}^{\infty} C_n \exp(-ian\pi/2) \phi_n(u).$$

Substituting for C_n and using the standard identity

$$\sum_{n=0}^{\infty} \exp(-in\alpha)\phi_n(u)\phi_n(u') = \sqrt{1 - i\cot\alpha}\, \exp[i\pi(\cot\alpha\; u^2 - 2\csc\alpha\; uu' + \cot\alpha\; u'^2)],$$

we obtain

$$\begin{aligned} f_a(u) &= \sqrt{1 - i\cot(a\pi/2)} \\ &\quad\times \int \mathrm{e}^{i\pi(\cot(a\pi/2)\, u^2 - 2\csc(a\pi/2)\, uu' + \cot(a\pi/2)\, u'^2)}\, f(u')\, \mathrm{d}u', \end{aligned} \tag{2}$$

from which the fractional Fourier transform of any function can be obtained by integration.

The zeroth-order fractional Fourier transform operation is equal to the identity operation $\mathcal{I}$ and the first-order fractional Fourier transform operator $\mathcal{F}^1$ is equal to the ordinary Fourier transform operator. Integer values of a simply correspond to repeated application of the ordinary Fourier transform and negative integer values correspond to repeated application of the inverse ordinary Fourier transform. For instance, $\mathcal{F}^2$ corresponds to the Fourier transform of the Fourier transform and $\mathcal{F}^{-2}$ corresponds to the inverse Fourier transform of the inverse Fourier transform. The fractional Fourier transform operator satisfies index additivity so that the a_2th-order transform of the a_1th-order transform is equal to the $(a_1 + a_2)$th-order transform: $\mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_2 + a_1}$. The order a can be any real value, however the operator $\mathcal{F}^a$ is periodic in a with period 4 since $\mathcal{F}^2$ equals the parity operator $\mathcal{P}$ which maps $f(u)$ to $f(-u)$ and $\mathcal{F}^4$ equals the identity operator. Consequently, the range of a is usually restricted to $(-2, 2]$ or $[0, 4)$.

The fractional Fourier transform is related to several concepts appearing in diverse contexts. It has a potential application in every field where the ordinary Fourier transform and related techniques are used. One of these is wave and beam propagation and diffraction, which are the most basic physical embodiments of fractional Fourier transformation. In many different contexts, it is possible to interpret the solution of the wave equation and the propagation of waves as an act of continuously unfolding fractional Fourier transformation. Intimately related to wave motion is simple harmonic motion, whose relationship to the fractional Fourier transform we will discuss in detail in this paper. The harmonic oscillator is of central importance in classical and quantum mechanics not only because it represents the simplest and most basic vibrational system, but also because systems disturbed from equilibrium, no matter what the underlying forces are, can often be satisfactorily modeled in terms of harmonic oscillations.

The kernel of the fractional Fourier transform corresponds to the Green's function associated with the quantum-mechanical harmonic oscillator differential equation (Schrödinger's equation). It is known that harmonic oscillations correspond to circular (or elliptic) motions in the space-frequency plane (or phase space), and so does fractional Fourier transformation, with the ordinary Fourier transformation corresponding to a $\pi/2$ rotation.

The above statements will be discussed in detail throughout this paper. We will first review simple harmonic oscillation in its classical formulation. We will then briefly discuss the quantum-mechanical harmonic oscillator and show that the solution can be expressed in terms of the fractional Fourier transform. The rest of the paper will focus on the properties of the fractional Fourier transform, as well as presenting the transform of some common functions. The phase-space rotation and projection properties of the transform are of particular importance from the perspective of harmonic oscillation. The fractional Fourier transform has already found many applications in the areas of signal processing and communications. We hope that this paper will motivate the exploration of new applications in areas of interest to readers of this journal. Those interested in learning more are referred to a recent book on the subject [1] or the chapter-length treatment [2].

2. Simple Harmonic Oscillation

The simple harmonic oscillator is of central importance for both practical and theoretical purposes as a first approximation for vibrating systems. Systems vibrating with small amplitude about an equilibrium point can often be modeled and described as a simple harmonic oscillator. Although simple harmonic oscillation can also be discussed in the context of electrical, acoustical, or other physical modalities, we will base our discussion on mechanical systems.

2.1. Classical Analysis

We consider a simple mechanical oscillator characterized by a linear restoring force. In one dimension such a force can be represented as $F(x) = -\kappa x$ where κ is the restoring force constant and x is the displacement from equilibrium. According to Newton's second law the sum of all external forces F is the cause of change in momentum $p = mv$ as $F = \mathrm{d}p/\mathrm{d}t$. Thus $\mathrm{d}(mv)/\mathrm{d}t = -\kappa x$ and using the definition of velocity $v = \mathrm{d}x/\mathrm{d}t$,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\kappa}{m}x = 0. \tag{3}$$

This is the classical harmonic oscillator differential equation [3]. It is a second-order, ordinary, linear differential equation with constant coefficients so that the solution $x(t)$ has two constants of integration that are determined by the initial conditions, usually given by the initial value of displacement $x(0)$ and velocity $v(0)$. The explicit solution is given by:

$$x(t) = A\cos(2\pi f_0 t + \theta), \quad v(t) = -2\pi f_0 A \sin(2\pi f_0 t + \theta), \tag{4}$$

where

$$f_0 = \sqrt{\kappa/m}/2\pi, \quad A = \sqrt{x^2(0) + v^2(0)/4\pi^2 f_0^2} \quad \text{and} \quad \theta = \arctan(-v(0)/2\pi f_0 x(0)).$$

Sometimes it is more convenient to deal with displacement x and momentum $p = mv$, rather than velocity v. Time can be eliminated from the above two equations, where it appears as a parameter, to obtain $p(x)$; that is, momentum as a function of displacement. This can be achieved by squaring the two expressions in Equation (4) and adding them:

$$\frac{p^2}{m^2 4\pi^2 f_0^2 A^2} + \frac{x^2}{A^2} = 1. \tag{5}$$

The final expression is in the form of the equation of an ellipse. The ellipse associated with a one-dimensional oscillator lives in the two-dimensional space spanned by x and p, called *phase space*. The phase-space vector $[x(t), p(t)]^{\mathrm{T}}$ representing the state of the harmonic oscillator traces this ellipsoidal trajectory, with time as a parameter.

If we multiply both sides of Equation (5) by $m4\pi^2 f_0^2 A^2/2$ we obtain:

$$\frac{p^2}{2m} + \frac{1}{2} m 4\pi^2 f_0^2 x^2 = \frac{1}{2} m 4\pi^2 f_0^2 A^2. \tag{6}$$

The right-hand side is equal to the total energy. The left-hand side is known as the classical Hamiltonian $H(x, p)$, which can be interpreted as the sum of kinetic and potential energies. By appropriately scaling the displacement and momentum, the ellipse can be transformed into a circle. To provide an alternative perspective, we will perform this scaling on Hamiltons's canonical equations [4]:

$$\frac{\partial H}{\partial x} = -\frac{\mathrm{d}p}{\mathrm{d}t}, \quad \frac{\partial H}{\partial p} = \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{7}$$

These first-order equations constitute an alternative statement of the law of motion. Specializing these equations to the harmonic oscillator Hamiltonian $H(x, p)$ we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -m4\pi^2 f_0^2 x = -\kappa x, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p}{m}. \tag{8}$$

These two equations can be combined to obtain Equation (3). Rather, we first introduce the new variables $x = Au$ and $p = \sqrt{\kappa m}\, A\mu = 2\pi f_0 m A\mu$ where A is a constant with dimension of length. In terms of these new variables, the above equations can be rewritten as:

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = -2\pi f_0 u, \quad \frac{\mathrm{d}u}{\mathrm{d}t} = 2\pi f_0 \mu. \tag{9}$$

These equations define circular trajectories in phase space. Combining the two equations we obtain the second-order equation $\mathrm{d}^2u/\mathrm{d}t^2 + 4\pi^2 f_0^2 u = 0$. The reader may show easily that after eliminating time, the equation relating μ to u is that of a circle: $\mu^2 + u^2 = 1$. Harmonic oscillation corresponds to rotation in phase space.

2.2. Quantum Analysis

The simple harmonic oscillator has an exact and elegant, but not trivial solution in quantum mechanics. The 'equation of motion' in quantum mechanics is the Schrödinger equation given by

$$i2\pi h \frac{\partial \psi(x, t)}{\partial t} = \mathcal{H}\psi(x, t), \tag{10}$$

where $\psi(x, t)$ is the *wave function*, representing the quantum-mechanical solution of the problem. The quantum-mechanical Hamiltonian $\mathcal{H}$ for the harmonic oscillator is obtained from the classical Hamiltonian $p^2/2m+\kappa x^2/2 = p^2/2m+m4\pi^2 f^2x^2/2$ given in the previous subsection by replacing the classical momentum p by the operator $-ih\partial/\partial x$, where $h = 2\pi\hbar$ and $\hbar$ are two forms of Planck's constant. Schrödinger's equation now becomes

$$i\frac{h}{2\pi}\frac{\partial\psi(x,t)}{\partial t} = \left[-\frac{h^2}{8m\pi^2}\frac{\partial^2}{\partial x^2} + \frac{mf^2x^2}{2}\right]\psi(x,t). \tag{11}$$

By defining dimensionless parameters u and a through $u = x/\sqrt{h/mf}$ and $a\pi/2 = ft$, this equation can be put into the dimensionless form

$$\left[-\frac{1}{4\pi}\frac{\partial^2}{\partial u^2} + \pi u^2\right]\psi_a(u) = i\frac{2}{\pi}\frac{\partial\psi_a(u)}{\partial a}, \tag{12}$$

where u is a dimensionless variable corresponding to x, and a corresponds to t.

The solution to an equation of the form $C\psi_a(u) = (i2/\pi)\partial\psi_a(u)/\partial a$ is given by $\psi_a(u) = \exp(-i(a\pi/2)C)\psi_0(u)$. By formal analogy we can write the solution to Equation (12) as

$$\psi_a(u) = \mathrm{e}^{-i(a\pi/2)\left((-1/4\pi)\partial^2/\partial u^2+\pi u^2\right)}\psi_0(u), \tag{13}$$

where $\psi_0(u)$ serves as the initial or boundary condition. (Although we are not being rigorous here, the final results are nevertheless correct.) The right-hand side of Equation (13) is of the form of a hyper-differential operator [5]. We can show that $\psi_a(u)$ as given by Equation (13) is a solution by formal substitution and taking the partial derivative.

Alternatively, the solution to Equation (12) can be expressed as

$$\psi_a(u) = \int K_a(u,u')\psi_0(u')\,\mathrm{d}u'. \tag{14}$$

The kernel $K_a(u,u')$ is known as the Green's function for the system governed by the differential equation in question and is simply the response of the system to $\psi_0(u) = \delta(u-u')$. It can be shown by direct substitution that $K_a(u,u')$ is given by

$$K_a(u,u') = \mathrm{e}^{-ia\pi/4}A_\alpha \exp[i\pi(\cot\alpha\, u^2 - 2\csc\alpha\, uu' + \cot\alpha\, u'^2)],$$

$$A_\alpha \equiv \sqrt{1-i\cot\alpha}\,, \quad \alpha \equiv \frac{a\pi}{2}$$

when $a \neq 2l$ and $K_a(u,u') \equiv \mathrm{e}^{-il\pi}\delta(u-u')$ when $a = 4l$ and $K_a(u,u') \equiv \mathrm{e}^{-i(l\pi\pm 0.5\pi)}$ $\delta(u+u')$ when $a = 4l \pm 2$, where l is an integer. This kernel is very similar to the kernel of the ath-order fractional Fourier transform (Equation (2)) [6, 7, 9, 14, 15, 1, 2]. The only difference is the factor $\mathrm{e}^{-ia\pi/4}$, which is a consequence of the fact that the Hamiltonian associated with the fractional Fourier transform, which we denote by $\mathcal{H}_{\mathrm{FRT}}$, differs from the above Hamiltonian by the term $-1/2$:

$$\mathcal{H}_{\mathrm{FRT}} = -\frac{1}{4\pi}\frac{\partial^2}{\partial u^2} + \pi u^2 - \frac{1}{2}. \tag{15}$$

This small discrepancy is hardly ever an issue in both theory and applications. The hyperdifferential operator appearing on the right-hand side of Equation (13) can also be recognized as being similar to the ath-order fractional Fourier transform operator, differing only by the same factor $\mathrm{e}^{-ia\pi/4}$ [1].

In conclusion, we see that the time evolution of the wave function of a harmonic oscillator corresponds to continual fractional Fourier transformation. Mathematically, if $\psi_0(u)$ is the initial condition, the wave function $\psi_a(u)$ is given by

$$\psi_a(u) = \mathrm{e}^{-ia\pi/4}\mathcal{F}^a[\psi_0(u)](u), \tag{16}$$

where $u = x/\sqrt{h/mf}$, $a = 2ft/\pi$. The wave function at different instants of time is simply given by different-ordered fractional Fourier transforms of $\psi_0(u)$; in other words, the fractional Fourier transform is the time evolution operator of the harmonic oscillator.

To tie this to our discussion of the oscillator in classical terms, we consider the quantum-mechanical phase-space picture. Similar to the classical case, we will see that harmonic oscillation corresponds to circular rotation in quantum-mechanical phase space as well. To this end, we must first define a phase-space distribution for the wave function. The Wigner distribution is such a phase-space distribution defined for a function $\psi(u)$ as [17]:

$$W_\psi(u,\mu) = \int \psi(u+u'/2)\psi^*(u-u'/2)\,\mathrm{e}^{-i2\pi\mu u'}\,\mathrm{d}u'. \tag{17}$$

The Wigner distribution answers the question 'How much of the total energy is located near this position and momentum?' (Naturally, the answer to this question can only be given within limitations imposed by the uncertainty principle.) Three of the most important properties of the Wigner distribution are

$$\int W_\psi(u,\mu)\,\mathrm{d}\mu = \mathcal{R}_0[W_\psi(u,\mu)] = |\psi(u)|^2, \tag{18}$$

$$\int W_\psi(u,\mu)\,\mathrm{d}u = \mathcal{R}_{\pi/2}[W_\psi(u,\mu)] = |\Psi(\mu)|^2, \tag{19}$$

$$\int\!\!\int W_\psi(u,\mu)\,\mathrm{d}u\,\mathrm{d}\mu = \int |\psi(u)|^2\,\mathrm{d}u = \int |\Psi(\mu)|^2\,\mathrm{d}\mu \equiv \text{Energy}. \tag{20}$$

Here $\Psi(\mu)$ is the ordinary Fourier transform of $\psi(u)$ and $\mathcal{R}_\alpha$ denotes the integral projection (or Radon transform) operator which takes an integral projection of the two-dimensional function $W_\psi(u,\mu)$ onto an axis making angle α with the u axis, to produce a one-dimensional function. The 'Energy' is defined in Equation (20) in accordance with the conventions of signal processing as the square integral of the function; it may not correspond to physical energy or the mean value of a Hamiltonian. In fact, when $|\psi(u)|^2$ is a probability density as in quantum mechanics, it is equal to unity.

We already showed that time evolution of the wave function can be described as continual fractional Fourier transformation. One of the most important properties of the fractional Fourier transform states that the Wigner distribution of the ath-order fractional Fourier transform of a function is a clockwise rotated version of the Wigner distribution of the original function [10, 11]. Mathematically this can be stated as

$$W_{\psi_a}(u,\mu) = W_\psi(u\cos\alpha - \mu\sin\alpha, u\sin\alpha + \mu\cos\alpha)\alpha = \frac{a\pi}{2}. \tag{21}$$

Thus, as time passes and the wave function evolves through successively higher ordered fractional Fourier transforms, the Wigner distribution of the wave function rotates in phase space, in a completely analogous manner as the circular rotation in the classical case.

An immediate corollary of the above result, supported by Figure 1, is

$$\mathcal{R}_\alpha[W_\psi(u,\mu)] = |\psi_a(u)|^2, \tag{22}$$

which is a generalization of Equations (18) and (19). This equation means that the projection of the Wigner distribution of $\psi(u)$ onto the axis making angle α gives us $|\psi_a(u)|^2$, the squared magnitude of the ath fractional Fourier transform of the function. Since projection onto the u axis (the space domain) gives $|\psi(u)|^2$ and projection onto the μ axis (the momentum domain) gives $|\Psi(\mu)|^2$, it is natural to refer to the axis making angle α as the ath-order *fractional Fourier domain*.

The results presented above can be easily generalized to multiple-dimensional harmonic oscillators provided there is no coupling between the dimensions, or any such coupling can be neglected. The *separable* multi-dimensional fractional Fourier transform, which has been studied in [1, 12], appears in these results. The case where there is significant coupling between the dimensions has not been studied; however the nonseparable fractional Fourier transform discussed in [13] may be expected to play a role in this case.

The discrete harmonic oscillator and its relation to the discrete fractional Fourier transform is discussed in [16].

3. Properties of the Fractional Fourier Transform

In this section we will discuss some of the important properties of the fractional Fourier transform.

Linearity: Let $\mathcal{F}^a$ denote the ath-order fractional Fourier transform operator. Then

$$\mathcal{F}^a\left[\sum_k c_k\psi_k(u)\right] = \sum_k c_k[\mathcal{F}^a\psi_k(u)].$$

Integer orders: $\mathcal{F}^k = (\mathcal{F})^k$ where $\mathcal{F}$ denotes the ordinary Fourier transform operator. This property states that when a is equal to an integer k, the ath-order fractional Fourier transform is equivalent to the kth integer power of the ordinary Fourier transform, defined by repeated application. It also follows that $\mathcal{F}^2 = \mathcal{P}$ (the parity operator), $\mathcal{F}^3 = \mathcal{F}^{-1} = (\mathcal{F})^{-1}$ (the inverse transform operator), $\mathcal{F}^4 = \mathcal{F}^0 = \mathcal{I}$ (the identity operator), and $\mathcal{F}^j = \mathcal{F}^{j \bmod 4}$.

Inverse: $(\mathcal{F}^a)^{-1} = \mathcal{F}^{-a}$. In terms of the kernel, this property is stated as $K_a^{-1}(u,u') = K_{-a}(u,u')$.

Index additivity: $\mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_2+a_1}$. In terms of kernels this can be written as

$$K_{a_2+a_1}(u,u') = \int K_{a_2}(u,u'')K_{a_1}(u'',u')\,\mathrm{d}u''.$$

Commutativity: $\mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_1}\mathcal{F}^{a_2}$.

Associativity: $\mathcal{F}^{a_3}(\mathcal{F}^{a_2}\mathcal{F}^{a_1}) = (\mathcal{F}^{a_3}\mathcal{F}^{a_2})\mathcal{F}^{a_1}$.

Unitarity: $(\mathcal{F}^a)^{-1} = (\mathcal{F}^a)^{\mathrm{H}} = \mathcal{F}^{-a}$ where $()^{\mathrm{H}}$ denotes the conjugate transpose of the operator. In terms of the kernel, this property can stated as $K_a^{-1}(u,u') = K_a^*(u',u)$.

Eigenfunctions: The eigenfunctions of the fractional Fourier transform operator are Hermite–Gaussian functions

$$\phi_n(u) = (2^{1/4}/\sqrt{2^n n!})H_n(\sqrt{2\pi}\,u)\exp(-\pi u^2),$$

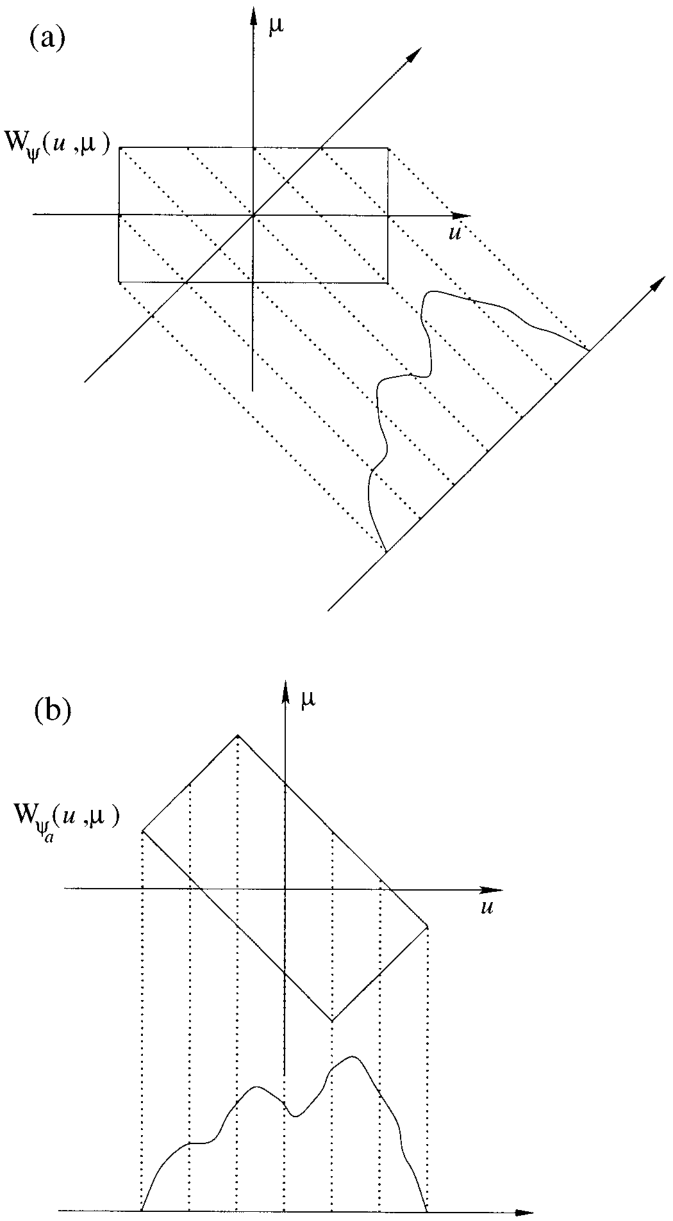

Figure 1. The integral projection of $W_{\psi}(u, \mu)$ onto the u_a axis which makes an angle $\alpha = a\pi/2$ with the u axis (a) is equal to the integral projection of $W_{\psi_a}(u, \mu)$ onto the u axis (b).

where $H_n(u)$ are the standard Hermite polynomials. The eigenvalue associated with the nth eigenfunction $\phi_n(u)$ is $\exp(-ina\pi/2)$.

Parseval: For any function $\psi(u)$ and $\upsilon(u)$,

$$\int \psi^*(u)\upsilon(u)\,\mathrm{d}u = \int \psi_a^*(u)\upsilon_a(u)\,\mathrm{d}u.$$

This property is equivalent to unitarity. Energy or norm conservation ($\mathrm{En}[\psi] = \mathrm{En}[\psi_a]$ or $\|\psi\| = \|\psi_a\|$) is a special case.

Time reversal: Let $\mathcal{P}$ denote the parity operator: $\mathcal{P}[\psi(u)] = \psi(-u)$, then

$$\mathcal{F}^a \mathcal{P} = \mathcal{P}\mathcal{F}^a, \tag{23}$$

$$\mathcal{F}^a[\psi(-u)] = \psi_a(-u). \tag{24}$$

Transform of a scaled function: Let $\mathcal{M}_M$ and $\mathcal{Q}_q$ denote the scaling $\mathcal{M}_M[\psi(u)] = |M|^{-1/2}\psi(u/M)$ and chirp multiplication $\mathcal{Q}_q[\psi(u)] = \mathrm{e}^{-i\pi q u^2}\psi(u)$ operators respectively. Then

$$\mathcal{F}^a\, \mathcal{M}_M = \mathcal{Q}_{[-\cot\alpha\left(1-(\cos^2\alpha')/(\cos^2\alpha)\right)]}\, \mathcal{M}_{[M\sin\alpha'/\sin\alpha]}\, \mathcal{F}^{a'}, \tag{25}$$

$$\mathcal{F}^a[|M|^{-1/2}\psi(u/M)] = \sqrt{\frac{1-i\cot\alpha}{1-iM^2\cot\alpha}} \times \mathrm{e}^{i\pi u^2\cot\alpha(1-(\cos^2\alpha')/(\cos^2\alpha))}\psi_{a'}\left(\frac{Mu\sin\alpha'}{\sin\alpha}\right). \tag{26}$$

Here $\alpha' = \arctan(M^{-2}\tan\alpha)$ and α' is taken to be in the same quadrant as α. This property is the generalization of the ordinary Fourier transform property stating that the Fourier transform of $\psi(u/M)$ is $|M|\Psi(M\mu)$. Notice that the fractional Fourier transform of $\psi(u/M)$ cannot be expressed as a scaled version of $\psi_a(u)$ for the same order a. Rather, the fractional Fourier transform of $\psi(u/M)$ turns out to be a scaled and chirp modulated version of $\psi_{a'}(u)$ where $a' \neq a$ is a different order.

Transform of a shifted function: Let $\mathcal{SH}_{u_0}$ and $\mathcal{PH}_{\mu_0}$ denote the shift $\mathcal{SH}_{u_0}[\psi(u)] = \psi(u-u_0)$ and the phase shift $\mathcal{PH}_{\mu_0}[\psi(u)] = \exp(i2\pi\mu_0 u)\psi(u)$ operators respectively. Then

$$\mathcal{F}^a\, \mathcal{SH}_{u_0} = \mathrm{e}^{i\pi u_0^2\sin\alpha\cos\alpha}\mathcal{PH}_{-u_0\sin\alpha}\,\mathcal{SH}_{u_0\cos\alpha}\mathcal{F}^a, \tag{27}$$

$$\mathcal{F}^a[\psi(u-u_0)] = \mathrm{e}^{i\pi u_0^2\sin\alpha\cos\alpha}\,\mathrm{e}^{-i2\pi u u_0\sin\alpha}\,\psi_a(u-u_0\cos\alpha). \tag{28}$$

We see that the $\mathcal{SH}_{u_0}$ operator, which simply results in a translation in the u domain, corresponds to a translation followed by a phase shift in the ath fractional domain. The amount of translation and phase shift is given by cosine and sine multipliers which can be interpreted in terms of ‘projections’ between the axes.

Transform of a phase-shifted function:

$$\mathcal{F}^a\, \mathcal{PH}_{\mu_0} = \mathrm{e}^{-i\pi\mu_0^2\sin\alpha\cos\alpha}\mathcal{PH}_{\mu_0\cos\alpha}\,\mathcal{SH}_{\mu_0\sin\alpha}\mathcal{F}^a, \tag{29}$$

$$\mathcal{F}^a[\psi(u-u_0)] = \mathrm{e}^{-i\pi\mu_0^2\sin\alpha\cos\alpha}\mathrm{e}^{i2\pi u\mu_0\cos\alpha}\,\psi_a(u-\mu_0\sin\alpha). \tag{30}$$

Similar to the shift operator, the phase-shift operator which simply results in a phase shift in the u domain, corresponds to a translation followed by a phase shift in the ath fractional domain. Again the amount of translation and phase shift are given by cosine and sine multipliers.

Transform of a coordinate multiplied function: Let $\mathcal{U}$ and $\mathcal{D}$ denote the coordinate multiplication $\mathcal{U}[\psi(u)] = u\psi(u)$ and differentiation $\mathcal{D}[\psi(u)] = (i2\pi)^{-1}\,\mathrm{d}\psi(u)/\mathrm{d}u$ operators respectively. Then

$$\mathcal{F}^a\,\mathcal{U}^n = [\cos\alpha\ \mathcal{U} - \sin\alpha\ \mathcal{D}]^n\mathcal{F}^a, \tag{31}$$

$$\mathcal{F}^a[u^n\psi(u)] = [\cos\alpha\ u - \sin\alpha\ (i2\pi)^{-1}\mathrm{d}/\mathrm{d}u]^n\psi_a(u). \tag{32}$$

When $a = 1$ the transform of a coordinate multiplied function $u\psi(u)$ is the derivative of the transform of the original function $\psi(u)$, a well-known property of the Fourier transform. For arbitrary values of a, we see that the transform of $u\psi(u)$ is a linear combination of the coordinate-multiplied transform of the original function and the derivative of the transform of the original function. The coefficients in the linear combination are $\cos\alpha$ and $-\sin\alpha$. As a approaches 0, there is more $u\psi(u)$ and less $\mathrm{d}\psi(u)/\mathrm{d}u$ in the linear combination. As a approaches 1, there is more $\mathrm{d}\psi(u)/\mathrm{d}u$ and less $u\psi(u)$.

Transform of the derivative of a function:

$$\mathcal{F}^a\,\mathcal{D}^n = [\sin\alpha\ \mathcal{U} + \cos\alpha\ \mathcal{D}]^n\mathcal{F}^a, \tag{33}$$

$$\mathcal{F}^a[[(i2\pi)^{-1}\mathrm{d}/\mathrm{d}u]^n\psi(u)] = [\sin\alpha\ u + \cos\alpha\ (i2\pi)^{-1}\mathrm{d}/\mathrm{d}u]^n\psi_a(u). \tag{34}$$

When $a = 1$ the transform of the derivative of a function $\mathrm{d}\psi(u)/\mathrm{d}u$ is the coordinate-multiplied transform of the original function. For arbitrary values of a, we see that the transform is again a linear combination of the coordinate-multiplied transform of the original function and the derivative of the transform of the original function.

Transform of a coordinate divided function:

$$\mathcal{F}^a[\psi(u)/u] = -i\csc\alpha\,\mathrm{e}^{i\pi u^2\cot\alpha}\int_{-\infty}^{2\pi u}\psi_a(u')\,\mathrm{e}^{(-i\pi u'^2\cot\alpha)}\,\mathrm{d}u'. \tag{35}$$

Transform of the integral of a function:

$$\mathcal{F}^a\left[\int_{u_0}^{u}\psi(u')\,\mathrm{d}u'\right] = \sec\alpha\,\mathrm{e}^{-i\pi u^2\tan\alpha}\int_{u_0}^{u}\psi_a(u')\mathrm{e}^{i\pi u'^2\tan\alpha}\,\mathrm{d}u'. \tag{36}$$

A few additional properties are

$$\mathcal{F}^a[\psi^*(u)] = \psi^*_{-a}(u), \tag{37}$$

$$\mathcal{F}^a[(\psi(u) + \psi(-u))/2] = (\psi_a(u) + \psi_a(-u))/2, \tag{38}$$

$$\mathcal{F}^a[(\psi(u) - \psi(-u))/2] = (\psi_a(u) - \psi_a(-u))/2. \tag{39}$$

It is also possible to write convolution and multiplication properties for the fractional Fourier transform, though these are not of great simplicity [1].

We may finally note that the transform is continuous in the order a. That is, small changes in the order a correspond to small changes in the transform $\psi_a(u)$. Nevertheless, care is always

Table 1. Transform pairs of some common functions.

Original Function	ath-order fractional Fourier transform
1	$\sqrt{1+i\tan\alpha}\,\mathrm{e}^{-i\pi u^2\tan\alpha}$ This equation is valid when $a \neq 2j+1$ where j is an arbitrary integer. The transform is $\delta(u)$ when $a = 2j+1$.
$\delta(u-u_0)$	$\sqrt{1-i\cot\alpha}\,\mathrm{e}^{i\pi(u^2\cot\alpha-2uu_0\csc\alpha+u_0^2\cot\alpha)}$. This expression is valid when $a \neq 2j$. The transform of $\delta(u-u_0)$ is $\delta(u-u_0)$ when $a = 4j$ and $\delta(u+u_0)$ when $a = 4j+2$.
$\exp(i2\pi\mu_0 u)$	$\sqrt{1+i\tan\alpha}\,\mathrm{e}^{-i\pi(u^2\tan\alpha-2u\mu_0\sec\alpha+\mu_0^2\tan\alpha)}$. This equation is valid when $a \neq 2j+1$. The transform of $\exp(i2\pi\mu_0 u)$ is $\delta(u-\mu_0)$ when $a = 4j+1$ and $\delta(u+\mu_0)$ when $a = 4j+3$.
$\exp[i\pi(\chi u^2 + 2\xi u)]$	$\sqrt{\frac{1+i\tan\alpha}{1+\chi\tan\alpha}}\,\mathrm{e}^{i\pi[u^2(\chi-\tan\alpha)+2u\xi\sec\alpha-\xi^2\tan\alpha]/[1+\chi\tan\alpha]}$ This equation is valid when $a-(2/\pi)\arctan\chi \neq 2j+1$. The transform of $\exp(i\pi\chi u^2)$ is $\sqrt{1/(1-i\chi)}\,\delta(u)$ when $[a-(2/\pi)\arctan\chi] = 2j+1$ and $\sqrt{1/(1-i\chi)}$ when $[a-(2/\pi)\arctan\chi] = 2j$.
$\phi_n(u)$	$\mathrm{e}^{-in\alpha}\,\phi_n(u)$.
$\exp[-\pi(\chi u^2 + 2\xi u)]$	$\sqrt{\frac{1-i\cot\alpha}{\chi-i\cot\alpha}}\,\mathrm{e}^{i\pi\cot\alpha[u^2(\chi^2-1)+2u\chi\xi\sec\alpha+\xi^2]/[\chi^2+\cot\alpha]}$ $\times\,\mathrm{e}^{-\pi\csc^2\alpha(u^2\chi+2u\xi\cos\alpha-\chi\xi^2\sin^2\alpha)/(\chi^2+\cot\alpha)}$. Here $\chi > 0$ is required for convergence.

required in dealing with cases where a approaches an even integer, since in this case the kernel approaches a delta function.

4. Transforms of Some Common Functions

We list the fractional Fourier transforms of some common functions in Table 1. Transforms of most other functions cannot usually be calculated analytically and therefore they must be computed numerically. As with the ordinary Fourier transform there is an order of $N\log N$ algorithm to compute the fractional transform of a continuous function whose time- or space-bandwidth product is N [30]. Therefore any improvements that come with use of the fractional Fourier transform come at no additional cost. The pure discrete fractional Fourier transform has also been defined and studied in [1, 35].

Figure 2 shows the magnitude of the fractional Fourier transforms of the rectangle function for different values of the order $a \in [0, 1]$. We observe that as a varies from 0 to 1, the rectangle function evolves into a sinc function, which is the ordinary Fourier transform of the rectangle function.

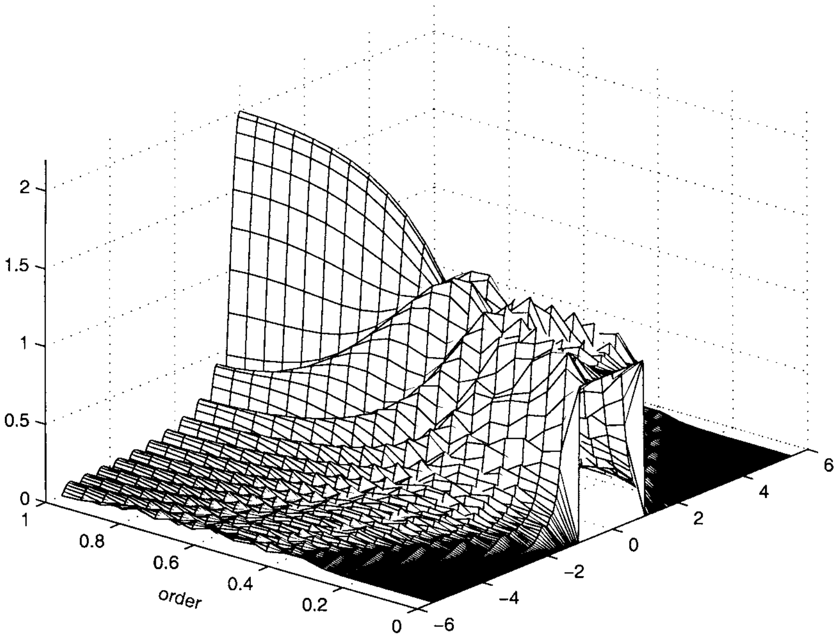

Figure 2. Magnitude of the fractional Fourier transform of a rectangle function as a function of the transform order.

5. Applications

In this paper we show how the fractional Fourier transform is intimately related to the harmonic oscillator in both its classical and quantum-mechanical forms. The kernel $K_a(u, u')$ of the fractional Fourier transform is essentially the Green's function (time-evolution operator kernel) of the quantum-mechanical harmonic oscillator differential equation. In other words, the time evolution of the wave function of a harmonic oscillator corresponds to continual fractional Fourier transformation. In classical mechanics, the relationship can be most easily seen by noting that – with properly normalized coordinates – harmonic oscillation corresponds to rotation in phase space. Just as the ensemble of particles of classical mechanics rotates in classical phase space, the Wigner distribution of the wave function in quantum mechanics rotates in quantum-mechanical phase space, a process which can be elegantly described by fractional Fourier transformation. Therefore, one can expect the fractional Fourier transform to play an important role in the study of vibrating systems, an application area which has so far not received attention.

Below we highlight some of the other applications of the fractional Fourier transform which have received the greatest interest so far. A more comprehensive treatment and an extensive list of references may be found in [1, 2].

The fractional Fourier transform has received a great deal of interest in the area of optics and especially optical signal processing (also known as Fourier optics or information optics) [18–21]. Optical signal processing is an analog signal processing method which relies on the representation of signals by light fields and their manipulation with optical elements such as lenses, prisms, transparencies, holograms and so forth. Its key component is the optical Fourier transformer which can be realized using one or two lenses separated by certain distances from the input and output planes. It has been shown that the fractional Fourier transform can be optically implemented with equal ease as the ordinary Fourier transform, allowing a generalization of conventional approaches and results to their more flexible or general fractional analogs.

The fractional Fourier transform has also been shown to be intimately related to wave and beam propagation and diffraction. The process of diffraction of light, or any other disturbance satisfying a similar wave equation, has been shown to be nothing but a process of continual fractional Fourier transformation; the distribution of light becomes fractional Fourier transformed as it propagates, evolving through continuously increasing orders.

The transform has also found widespread use in signal and image processing, in areas ranging from time/space-variant filtering, perspective projections, phase retrieval, image restoration, pattern recognition, tomography, data compression, encryption, watermarking, and so forth (for instance, [11, 22–29]). Concepts such as 'fractional convolution' and 'fractional correlation' have been studied. One of the most striking applications is that of filtering in fractional Fourier domains [22]. In traditional filtering, one takes the Fourier transform of a signal, multiplies it with a Fourier-domain transfer function, and inverse transforms the result. Here, we take the fractional Fourier transform, apply a filter function in the fractional Fourier domain, and inverse transform to the original domain. It has been shown that considerable improvement in performance is possible by exploiting the additional degree of freedom coming from the order parameter a. This improvement comes at no additional cost since computing the fractional Fourier transform is not more expensive than computing the ordinary Fourier transform [30]. The concept has been generalized to multi-stage and multi-channel filtering systems which employ several fractional Fourier domain filters of different orders [31, 32]. These schemes provide flexible and cost-efficient means of designing time/space-variant filtering systems to meet desired objectives and may find use in control systems. Another line of generalization is the use of the three-parameter family of *linear canonical transforms*, which may allow even further flexibility than the fractional Fourier transform [33].

Another potential application area is the solution of time-varying differential equations. Namias, McBride and Kerr [6, 7, 34] have shown how the fractional Fourier transform can be used to solve certain differential equations. Constant coefficient (time-invariant) equations can be solved with the ordinary Fourier or Laplace transforms. It has been shown that certain kinds of second-order differential equations with non-constant coefficients can be solved by exploiting the additional degree of freedom associated with the order parameter a. One proceeds by taking the fractional Fourier transform of the equation and then choosing a such that the second-order term disappears, leaving a first-order equation whose exact solution can always be written. Then, an inverse transform (of order $-a$) provides the solution of the original equation. It remains to be seen if this method can be generalized to higher-order equations by reducing the order from n to $n-1$ and proceeding recursively down to a first-order equation, by using a different-ordered transform at each step.

6. Conclusion

Not all vibrational systems exhibit simple harmonic motion, but the simple harmonic oscillator remains of central importance as the most basic vibrational system corresponding to the case of small disturbances from equilibrium. We have seen that the kernel of the fractional Fourier transform is the Green's function of the harmonic oscillator Schrödinger equation and that with passing time, the wave function undergoes continual fractional Fourier transformation.

These processes correspond to rotation in phase space. It is striking that whereas the solution of the quantum harmonic oscillator has been long well understood, it has not been appreciated that the time evolution operator is the fractional operator power of the Fourier transform. It is natural to expect that vibrational systems which depart from simple harmonic motion will have evolution operators which depart from the fractional Fourier transform. It would be of interest to study these perturbed transforms and their dependence on the perturbation parameters of the system. An interesting question is whether these perturbed transforms can be expressed as functions other than the simple fraction function $[\cdot]^a$ of the ordinary Fourier transform $\mathcal{F}$. If so, these perturbed operator functions would constitute interesting objects of study for the characterization of different kinds of perturbations. An alternative approach is suggested by examining the kernel of the fractional Fourier transform, which is of the quadratic exponential type. It may be possible to characterize perturbed systems by transforms whose kernels have other than quadratic terms in the exponent. These systems would exhibit non-circular trajectories in phase space. In conclusion, the close relationship between the fractional Fourier transform and simple harmonic motion may be just the beginning of new approaches and techniques for the study of more general vibrational systems.

Beyond harmonic oscillation and vibrating systems, we believe that the fractional Fourier transform is of potential usefulness in every area in which the ordinary Fourier transform is used. The typical pattern of discovery of a new application is to concentrate on an application where the ordinary Fourier transform is used and ask if any improvement or generalization might be possible by using the fractional Fourier transform instead. The additional order parameter often allows better performance or greater generality because it provides an additional degree of freedom over which to optimize.

Typically, improvements are observed or are greater when dealing with time/space-variant signals or systems. Furthermore, very large degrees of improvement often becomes possible when signals of a chirped nature or with nearly-linearly increasing frequencies are in question, since chirp signals are the basis functions associated with the fractional Fourier transform (just as harmonic functions are the basis functions associated with the ordinary Fourier transform).

The fractional Fourier transform has spurred interest in many other fractional transforms; see [1] for further references. The fractional Laplace and z-transforms, however, have so far not received sufficient attention. Development of these transforms are not only of interest in themselves, but should be instrumental in dealing with more general systems.

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