# Exact Relation Between Continuous and Discrete Linear Canonical Transforms

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***Abstract*—Linear canonical transforms (LCTs) are a family of integral transforms with wide application in optical, acoustical, electromagnetic, and other wave propagation problems. The Fourier and fractional Fourier transforms are special cases of LCTs. We present the exact relation between continuous and discrete LCTs (which generalizes the corresponding relation for Fourier transforms), and also express it in terms of a new definition of the discrete LCT (DLCT), which is independent of the sampling interval. This provides the foundation for approximately computing the samples of the LCT of a continuous signal with the DLCT. The DLCT in this letter is analogous to the DFT and approximates the continuous LCT in the same sense that the DFT approximates the continuous Fourier transform. We also define the bicanonical width product which is a generalization of the time-bandwidth product.**

***Index Terms*—Bicanonical width product, fractional Fourier transform, linear canonical series, linear canonical transform.**

## I. Introduction

DISCRETE counterparts of continuous transforms are not only of intrinsic interest, but are important for approximately computing the samples of continuous transforms. For instance, the discrete Fourier transform (DFT) is commonly used to obtain the samples of the Fourier transform (FT) of a function from the samples of the original function.

Linear canonical transforms (LCTs) are a three-parameter family of integral transforms with wide application in wave propagation problems [1] and have also found use in optimal filtering [2]. The Fourier and fractional Fourier transforms, coordinate scaling, and chirp multiplication and convolution operations, are special cases of LCTs. In this letter, we derive the exact relation between the continuous LCT and the discrete LCT (DLCT) defined in [3] and implemented in [4]. This provides the underlying foundation for approximately computing the samples of the LCT of a continuous signal by replacing the transform integral with a finite sum, and constitutes a generalization of the exact relation between continuous and discrete FTs, which has been regarded as a fundamental theorem by Papoulis [5]. Consequently, the DLCT in this letter approximates the continuous LCT in the same sense that the DFT approximates the continuous FT.

To state the above theorem for FTs, let $f(u)$ and $F(\mu)$ be a continuous-time signal and its FT, and define the periodically replicated functions $\bar{f}(u) \equiv \sum_{n=-\infty}^{\infty} f(u-n\Delta u)$ and $\bar{F}(\mu) \equiv \sum_{n=-\infty}^{\infty} F(\mu - n\Delta\mu)$, where $\Delta u$ and $\Delta\mu$ are arbitrary. Then, the samples of these functions form a DFT pair as follows [5]:

$$\bar{F}(m\delta\mu) = \delta u \sum_{k\in\langle N\rangle} \bar{f}(k\delta u) e^{-i2\pi mk/N} \tag{1}$$

for any $m$, where $\delta u = 1/\Delta\mu$, $\delta\mu = 1/\Delta u$, $N = \Delta u\Delta\mu$, and $\langle N\rangle$ denotes any interval of length $N$. This exact relation between the continuous and discrete Fourier transforms, provides the basis for approximately computing the samples of the continuous FT of a function by using the DFT.

In addition to generalizing the above fundamental theorem to LCTs, we also show that it can be expressed in terms of a new definition of the DLCT which, unlike certain earlier definitions, can be expressed without reference to the underlying continuous functions or their extents and sampling intervals. This new definition would be useful in purely discrete settings and in developing fast algorithms. In the process we define the linear canonical series, which is the generalization of the ordinary Fourier series. We also compare a computational algorithm based on these definitions of the DLCT, with earlier algorithms. Furthermore, we find an expression for the number of degrees of freedom of signals confined to finite intervals in the time and LCT domains. This result is significant since it constitutes a generalization of the time-bandwidth product. We refer to this new quantity as the *time-canonical width product* or more generally the *bicanonical width product*.

## II. Discrete Linear Canonical Transforms

The LCT with parameter matrix $\mathbf{M}$ is defined as [1]

$$f_{\mathbf{M}}(u) \equiv (\mathcal{C}_{\mathbf{M}}f)(u) \equiv \int_{-\infty}^{\infty} C_{\mathbf{M}}(u,u')f(u')du',$$

$$C_{\mathbf{M}}(u,u') \equiv \sqrt{\beta}e^{-i\pi/4}e^{i\pi(\alpha u^2 - 2\beta uu' + \gamma u'^2)} \tag{2}$$

where $\mathcal{C}_{\mathbf{M}}$ is the LCT operator, and $\alpha$, $\beta$, $\gamma$ are real parameters. The transform is unitary and $C_{\mathbf{M}}^{-1}(u,u') = C_{\mathbf{M}^{-1}}(u,u') = C_{\mathbf{M}}^{*}(u',u)$. The unit-determinant matrix $\mathbf{M}$ is equivalent to the three parameters and either set can be obtained from the other [1]: $\mathbf{M} \equiv [\gamma/\beta, 1/\beta; -\beta + \alpha\gamma/\beta, \alpha/\beta]$. The LCT reduces to the $a$th-order fractional Fourier transform (FRT) when $\alpha = \cot(a\pi/2)$, $\beta = \csc(a\pi/2)$, $\gamma = \cot(a\pi/2)$. The FRT operator $\mathcal{F}^a$ is additive in index: $\mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_2+a_1}$ and reduces to the FT and identity operators for $a=1$ and $a=0$.

The discrete LCT $\hat{f}_{\mathbf{M}}(m\delta u_{\mathbf{M}})$ of $\hat{f}(k\delta u)$ has been defined as follows for $m = -N/2, \ldots, N/2-1$ [3], [4]:

$$\hat{f}_{\mathbf{M}}(m\delta u_{\mathbf{M}}) \equiv \delta u \sum_{k=-N/2}^{N/2-1} \hat{f}(k\delta u) C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u),$$

$$C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u) = \sqrt{\beta}e^{-\frac{i\pi}{4}} e^{\frac{i\pi}{N|\beta|}(\alpha\frac{\delta u_{\mathbf{M}}}{\delta u}m^2 - 2\beta km + \gamma\frac{\delta u}{\delta u_{\mathbf{M}}}k^2)} \tag{3}$$

Manuscript received February 28, 2009; revised May 05, 2009. First published May 27, 2009; current version published June 12, 2009. The work of H. M. Ozaktas was supported in part by the Turkish Academy of Sciences. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Behrouz Farhang-Boroujeny.

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Digital Object Identifier 10.1109/LSP.2009.2023940

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where $\delta u_{\mathbf{M}} = (|\beta| N \delta u)^{-1}$. Here $\delta u$ and $\delta u_{\mathbf{M}}$ are the sampling intervals in the time and LCT domains. $N$ is the number of samples. The carets in (3) are to remind us that $\hat{f}_{\mathbf{M}}$ is not the continuous LCT of $\hat{f}$. The special case of (3) for the FRT has been defined in [6], but we note that this definition is different than the discrete FRT in [7]. The definition in (3) can be made unitary by including an additional factor $\sqrt{\delta u_{\mathbf{M}}/\delta u}$.

The definition in (3), while suitable for certain purposes, is not a usual way of defining a discrete transform, since the transform matrix exhibits the undesirable quality of depending on the sampling intervals, whereas ideally it would depend only on the number of samples $N$ and $\alpha$, $\beta$, $\gamma$. One of our contributions is to show that an interval-independent definition of the DLCT can still be used to approximately compute continuous LCTs with arbitrary sampling intervals.

We express the transform matrix of this interval-independent and unitary definition of the DLCT as follows:

$$\mathbf{C}_{\mathbf{M}}[m,k] = \frac{\sqrt{\beta} e^{-i\pi/4}}{\sqrt{N|\beta|}} e^{i\pi \frac{1}{N|\beta|}(\alpha m^2 - 2\beta k m + \gamma k^2)}. \quad (4)$$

This corresponds to the matrix elements in (3) with $\delta u = \delta u_{\mathbf{M}}$. We will demonstrate in Section III how to use this interval-independent DLCT to exactly compute DLCTs as defined in (3), as well as to approximately compute continuous LCTs.

## III. Fundamental Theorem for LCTs

Let $f(u)$ and $f_{\mathbf{M}}(u)$ be a continuous-time signal and its LCT with parameters $\alpha, \beta, \gamma$. Define the following periodically replicated functions where each period has been modulated with varying phase terms:

$$\bar{f}(u)_{(\mathbf{M}^{-1},\Delta u)} \equiv \sum_{n=-\infty}^{\infty} f(u - n\Delta u) \times e^{-i\pi\gamma n \Delta u (2u - n\Delta u)}, \quad (5)$$

$$\bar{f}_{\mathbf{M}}(u)_{(\mathbf{M},\Delta u_{\mathbf{M}})} \equiv \sum_{n=-\infty}^{\infty} f_{\mathbf{M}}(u - n\Delta u_{\mathbf{M}}) \times e^{i\pi\alpha n \Delta u_{\mathbf{M}} (2u - n\Delta u_{\mathbf{M}})} \quad (6)$$

where $\Delta u$ and $\Delta u_{\mathbf{M}}$ are arbitrary. Both definitions are of identical form since the value of $\alpha$ for $\mathbf{M}^{-1}$ is $-\gamma$ [1]. It is also worth noting that the functions we have just defined are chirp-periodic in the sense of [6], [8].

The generalization of the exact relation between continuous and discrete FTs (1) to LCTs will be stated as a theorem:

*Theorem:* The samples of the functions defined in (5) and (6) are exactly related to each other through the samples of the continuous kernel [the DLCT matrix in (3)]:

$$\bar{f}_{\mathbf{M}}(m\delta u_{\mathbf{M}})_{(\mathbf{M},\Delta u_{\mathbf{M}})} = \delta u \sum_{k\in\langle N\rangle} \bar{f}(k\delta u)_{(\mathbf{M}^{-1},\Delta u)} C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u) \quad (7)$$

for any $m$, where

$$\delta u = \frac{1}{|\beta|\Delta u_{\mathbf{M}}}, \quad \delta u_{\mathbf{M}} = \frac{1}{|\beta|\Delta u}, \quad N = \Delta u \Delta u_{\mathbf{M}} |\beta|. \quad (8)$$

Postponing the proof, we also express this exact relation in terms of the interval-independent DLCT as a *Corollary*:

$$\bar{f}_{\mathbf{M}}(m\delta u_{\mathbf{M}})_{(\mathbf{M},\Delta u_{\mathbf{M}})} = \sqrt{\frac{\Delta u}{\Delta u_{\mathbf{M}}}} \sum_{k\in\langle N\rangle} \bar{f}(k\delta u)_{(\mathbf{M}^{-1},\Delta u)} \mathbf{C}_{\mathbf{M}'}[m,k] \quad (9)$$

where $\mathbf{M}'$ corresponds to $\alpha' = \alpha\Delta u_{\mathbf{M}}/\Delta u$, $\beta' = \beta$, $\gamma' = \gamma\Delta u/\Delta u_{\mathbf{M}}$. Thus, the interval-independent DLCT defined in (4) exactly relates the samples of the functions defined in (5) and (6) to each other. The parameters $\alpha'$, $\beta'$, $\gamma'$ differ from the original $\alpha$, $\beta$, $\gamma$ because using the interval-independent DLCT effectively involves a rescaling of the sampling intervals, and the LCT of a scaled version of a function, is a scaled version of the LCT of the original function with different parameters.

The definition of the functions in (5) and (6), and the theorem and corollary can easily be specialized to the FRT by replacing $\alpha \to \cot(a\pi/2)$, $\beta \to \csc(a\pi/2)$, $\gamma \to \cot(a\pi/2)$.

*Proof of Theorem:* Let $f_s(u)$ be the sampled version of a continuous signal $f(u)$ with sampling interval $\delta u$:

$$f_s(u) = \sum_{n=-\infty}^{\infty} f(n\delta u)\delta(u - n\delta u) = \frac{1}{\delta u}\sum_{n=-\infty}^{\infty} f(u) e^{i2\pi n u/\delta u}. \quad (10)$$

Then, apply the LCT operator $\mathcal{C}_{\mathbf{M}}$ to the equivalent expressions for $f_s(u)$ in (10) to obtain

$$\bar{f}_{\mathbf{M}}(u)_{(\mathbf{M},\Delta u_{\mathbf{M}})} = \delta u \sum_{n=-\infty}^{\infty} f(n\delta u) C_{\mathbf{M}}(u, n\delta u) \quad (11)$$

where $\Delta u_{\mathbf{M}} = (|\beta|\delta u)^{-1}$. This result is the generalization of the *Poisson sum formula* [5], and is related to the LCT sampling theorem [9]–[11]. The right-hand side of this expression defines the *discrete-time* LCT [4] and its special case for the FRT defines the *discrete-time* FRT [6].

Now, sample $\bar{f}_{\mathbf{M}}(u)_{(\mathbf{M},\Delta u_{\mathbf{M}})}$ in (11) with a sampling interval chosen as $\delta u_{\mathbf{M}} = (|\beta| N \delta u)^{-1}$ with $N$ an arbitrary integer. Then write the integer $n$ as $n = k + rN$, $k \in \langle N\rangle$, where $r$ is an integer running from $-\infty$ to $\infty$:

$$\bar{f}_{\mathbf{M}}(m\delta u_{\mathbf{M}})_{(\mathbf{M},\Delta u_{\mathbf{M}})} = \delta u \sum_{r=-\infty}^{\infty} \sum_{k\in\langle N\rangle} \times f((k+rN)\delta u) C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, (k+rN)\delta u). \quad (12)$$

After changing the order of summations in the above equation and substituting $C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, (k+rN)\delta u) = C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u)\, e^{i\pi\gamma r N \delta u^2 (2k + rN)}$, we collect all the terms that depend on $r$ in a summation and recognize this summation as the sampled version of (5) with the sampling interval $\delta u$ where $\Delta u = N\delta u$. This completes the proof of (7). □

*Proof of Corollary:* Substitute $\alpha'$, $\beta'$, $\gamma'$ for $\alpha$, $\beta$, $\gamma$ in (4). Use (8) and the DLCT matrix in (3) to obtain $\mathbf{C}_{\mathbf{M}'}[m,k] = \sqrt{\delta u \delta u_{\mathbf{M}}} C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u)$. Substitute this in (7). □

Had the proof of the theorem been carried out by applying the operator $\mathcal{C}_{\mathbf{M}}^{-1}$ to the sampled version of $f_{\mathbf{M}}(u)$ instead of applying $\mathcal{C}_{\mathbf{M}}$ to the sampled version of $f(u)$, one would obtain the duals of (7) and (11):

$$\bar{f}(k\delta u)_{(\mathbf{M}^{-1},\Delta u)} = \delta u_{\mathbf{M}} \sum_{m\in\langle N\rangle} \bar{f}_{\mathbf{M}}(m\delta u_{\mathbf{M}})_{(\mathbf{M},\Delta u_{\mathbf{M}})} C^*_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u) \quad (13)$$

$$\bar{f}(u)_{(\mathbf{M}^{-1},\Delta u)} = \delta u_{\mathbf{M}} \sum_{n=-\infty}^{\infty} f_{\mathbf{M}}(n\delta u_{\mathbf{M}}) C^*_{\mathbf{M}}(n\delta u_{\mathbf{M}}, u). \quad (14)$$

Here (13) provides the exact relation for the inverse DLCT and (14) gives the expression for the linear canonical series, which

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generalizes ordinary Fourier series. The fractional Fourier series in [6], [12], [13] is a special case of this series. Just as periodic functions have Fourier series, a function in the form of (5), which is chirp-periodic, has a linear canonical series. Here, the series coefficients of $\bar{f}(u)_{(\mathbf{M}^{-1},\Delta u)}$ are $\delta u_{\mathbf{M}} f_{\mathbf{M}}(n\delta u_{\mathbf{M}})$. Again in analogy with the ordinary Fourier case, linear canonical series can also be used to represent an aperiodic signal $f(u)$ with finite extent. In this case, the series will give the periodically replicated and phase modulated extension of $f(u)$ outside its finite extent. Unlike the *discrete-time* LCTs, which take discrete signals to continuous signals [4], [6], [12], linear canonical series, which take continuous signals to discrete signals, do not seem to have received attention in the literature.

Just as periodicity and discreteness in either the time or frequency domain implies the dual property in the other domain [14], (11) and (14) show that discreteness in either the time or LCT domain implies periodic replication and phase modulation in the other domain, and vice versa [6]. If both are present in one domain, they will both also be present in the other domain. It is precisely in this case that, there exists an exact relation between these two sets of samples ( (7) and (13)).

## IV. Computation of Continuous LCTs

The exact relation between continuous and discrete LCTs provides the underlying foundation for approximately computing the samples of the LCT of a continuous signal by replacing the transform integral with a finite sum. Sampling the continuous input function and the transform kernel will always lead to a finite sum; however, this sum will not be exactly equal to the samples of the continuous output. We may still choose this finite sum as the definition of the discrete version of our transform, but then the relationship between the discrete input and output vectors, and the samples of the continuous input and output remains to be shown. In particular, for the DLCT in (3), the relation of $\hat{f}$ and $\hat{f}_{\mathbf{M}}$ with the samples of the original continuous functions is not apparent and our main contribution is to exactly provide this relation ( (7) and (9)).

Let us assume that a large percentage of the total energy of the signal is respectively concentrated in the intervals $[-\Delta u/2, \Delta u/2]$ and $[-\Delta u_{\mathbf{M}}/2, \Delta u_{\mathbf{M}}/2]$ in the time and LCT domains. Then, $\bar{f}(u)_{(\mathbf{M}^{-1},\Delta u)} \approx f(u)$ and $\bar{f}_{\mathbf{M}}(u)_{(\mathbf{M},\Delta u_{\mathbf{M}})} \approx f_{\mathbf{M}}(u)$ in the respective intervals, and from (7) and (9) the discrete LCT of the samples of the function are the approximate samples of the continuous LCT of that function:

$$f_{\mathbf{M}}(m\delta u_{\mathbf{M}}) \approx \delta u \sum_{k=-N/2}^{N/2-1} f(k\delta u) C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u) \qquad (15)$$

$$f_{\mathbf{M}}(m\delta u_{\mathbf{M}}) \approx \sqrt{\frac{\Delta u}{\Delta u_{\mathbf{M}}}} \sum_{k=-N/2}^{N/2-1} f(k\delta u) \mathbf{C}_{\mathbf{M}'}[m,k] \qquad (16)$$

where $\delta u$, $\delta u_{\mathbf{M}}$, and $N$ are as in (8). If both the functions $f(u)$ and $f_{\mathbf{M}}(u)$ could be identically zero outside of the given intervals, the mapping between the samples of these functions would be exact. But, since the extent of a function and its LCT cannot both be finite for $\beta \neq \infty$ [15], there will be overlaps between the periodically replicated and phase modulated functions, and the DLCT will be an approximation between the samples of the continuous signals. This approximation for the LCT and FRT is similar to that for the FT. The functions (5) and (6) reveal the precise nature of overlap and aliasing that occurs, which is somewhat different than the Fourier case due to the phase terms appearing in the periodic replication. As with the DFT, the approximation improves with increasing $N$ since this decreases the overlap between the replicas.

As is well-known, if the time-domain vector is periodic or periodically extended, the DFT summation can run over any interval of length $N$; furthermore, the output vector is periodic with period $N$. Likewise, if the time-domain vector is chirp-periodic or chirp-periodically extended (as in (5)), then the DLCT summation can run over any interval of length $N$; furthermore, the output vector is chirp-periodic (as in (6)).

Both the DLCT in (3) and the interval-independent DLCT whose matrix is given in (4) can be computed by performing a chirp multiplication, a fast Fourier transform (FFT) and a second chirp multiplication, which takes $2N + (N/2)\log N$ time, where $N = \Delta u \Delta u_{\mathbf{M}} |\beta|$ [3], [6]. It is interesting to compare this approach to computing LCTs with the algorithms given in [16]–[18]. All of these produce output vectors which are good approximations to the samples of the continuous transform, limited only by the fundamental fact that a signal cannot have finite extent in more than one domain; since the sampling interval is ensured to satisfy the Nyquist criterion, the output samples can be used to reconstruct good approximations of the continuous output. On the other hand, while the algorithms in [16], [17] also take $\sim N \log N$ time, most of them involve more than one FFT and therefore a larger factor in front, in addition to being less transparent. However, this does not automatically mean that these earlier algorithms are slower since the number of samples $N$ in these works are not directly comparable to that in this letter, as discussed below.

## V. Generalization of the Time-Bandwidth Product

The conventional time-bandwidth product $\Delta u \Delta \mu$ is the minimum number of samples to identify a signal out of all signals whose energies are confined to time and frequency intervals of length $\Delta u$ and $\Delta \mu$. Likewise, the product $\Delta u \Delta u_{\mathbf{M}} |\beta|$ is the minimum number of samples to identify a signal out of all signals whose energies are confined to time and LCT intervals of length $\Delta u$ and $\Delta u_{\mathbf{M}}$. We refer to $\Delta u \Delta u_{\mathbf{M}} |\beta|$ as the *time-canonical width product*. More generally, the term *bi-canonical width product* will be used to refer to the product $\Delta u_{\mathbf{M}_1} \Delta u_{\mathbf{M}_2} |\beta_{12}|$, where $\Delta u_{\mathbf{M}_1}$ and $\Delta u_{\mathbf{M}_2}$ are the extents of the signal in two arbitrary LCT domains and $\beta_{12}$ is the parameter of the LCT between these domains. The minimum number of samples to uniquely identify a signal is also referred to as the number of degrees of freedom.

The time-bandwidth product is a notion derived from simultaneously specifying the time and frequency extents of signals. Although this is commonly seen as an intrinsic property, it is in fact a notion that is specific to the frequency domain. However, it is always possible to specify the extent of a signal in other FRT or LCT domains. The set of signals thus specified will constitute a different family of signals with a different number of degrees of freedom than that defined through specifying the extent in the frequency domain. Indeed, there is no reason to think that families of signals encountered in practice will necessarily uniformly fall into a rectangular region in the time-frequency space. For instance, in applications where the underlying physics involves LCT type integrals as is the case with propagation problems, specification of $\Delta u$ and $\Delta u_{\mathbf{M}}$ may provide a better fit to the set of signals we are dealing with.

While having a finite extent in one LCT domain is not sufficient to ensure that a family of signals has a finite number of degrees of freedom, specifying two LCT domains in which

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the signal is approximately confined to finite intervals allows us to approximately represent the family of signals with a finite number of degrees of freedom. The family of signals thus defined depends both on the chosen LCT domains and the extent of the signals in those domains.

To approximately compute LCTs, we assume that the signal is approximately confined to $\Delta u$ and $\Delta u_{\mathbf{M}}$ in the time and LCT domains. In contrast, in [16], [17] it is assumed that the signal is confined to a rectangle or ellipse orthogonal to the time-frequency axes in the time-frequency plane, regardless of the parameters of the FRT or LCT to be computed. As noted, it is not possible to directly compare the present algorithm to those in [17] since different families of signals are assumed. Therefore, which algorithm is better will depend strongly on what assumptions are best suited to the family of signals we are dealing with. However, if we focus on [16] which deals with the special case of FRTs, a comparison becomes possible. There the signal is assumed to have negligible energy outside a circle of diameter $\Delta u$ in the time-frequency plane. This implies that the signal will be approximately confined to $\Delta u$ in both the time and FRT domains [19], so that the results of this letter can be applied. The value of $N = \Delta u^2 |\csc(a\pi/2)|$ in our complexity expressions is smaller than $N = 2\Delta u^2$ appearing in [16] for $0.5 \leq |a| \leq 1.5$, but the real advantage lies in the fact that the numerical factor in front of $N \log N$ will be considerably smaller than in this widely-used method.

It is interesting to note that the relations between the parameters given in (8) are consistent with sampling theorems for the FRT [6], [13], [20], [21] and LCT [9]–[11], as well as with the bicanonical width product. In (8), $\delta u^{-1} = |\beta| \Delta u_{\mathbf{M}}$ is the minimum rate for sampling the time-domain representation of a signal that has finite extent $\Delta u_{\mathbf{M}}$ in the LCT domain in question. If we sample the time-domain signal at this rate, the total number of samples over the extent $\Delta u$ is given by $N = \Delta u / \delta u = \Delta u \Delta u_{\mathbf{M}} |\beta|$, which is the same as the number of samples $N$ given in (8), and nothing but the bicanonical width product. Alternatively, $\delta u_{\mathbf{M}}^{-1} = |\beta| \Delta u$ in (8) is the minimum rate for sampling the LCT-domain representation of a signal that has finite extent $\Delta u$ in the time domain. If we sample the LCT-domain signal at this rate, the total number of samples over the extent $\Delta u_{\mathbf{M}}$ is once again given by $N = \Delta u_{\mathbf{M}} / \delta u_{\mathbf{M}} = \Delta u \Delta u_{\mathbf{M}} |\beta|$. Thus we have accomplished to formulate such that the number of samples in both domains are equal to each other regardless of the LCT parameters, and this number of samples is the minimum possible for both domains, for the given extents. This simple approach is in contrast to some earlier works where the starting assumption is knowledge of the extent of the signal in the time and frequency domains and the number of samples is determined from the ordinary Nyquist sampling theorem [17], [22], whereas in our formulation it is knowledge of the extents in two LCT domains and the number of samples is determined from the LCT sampling theorem. We also note that the relations in (8) reduce to the well-known results for the FT when $\beta = 1$.

As a final remark, we note that the relation between the extents of the signals and the number of samples expressed as $\Delta u \Delta u_{\mathbf{M}} = N / |\beta|$ is in agreement with the uncertainty relation for LCTs. Since $N \geq 1$ we can write $\Delta u \Delta u_{\mathbf{M}} \geq 1/|\beta|$ which is precisely the uncertainty relation for LCTs [1].

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