Equivalence Principle and Scattering from PEC Objects

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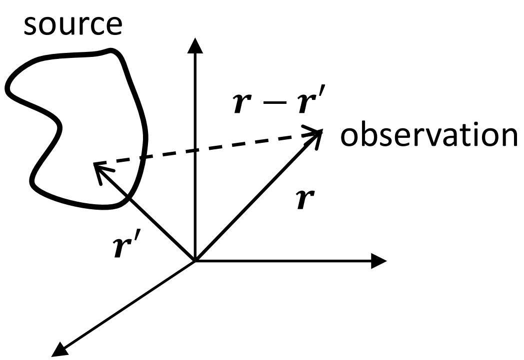
16 January 2015

From Sources to Fields: Integral Equations

Single-frequency sinsoidal time variation: $\boldsymbol{E}(\boldsymbol{r},t) = Re\{\boldsymbol{E}(\boldsymbol{r})e^{-i\omega t}\}$

Homogenious and isotropic medium: $(\boldsymbol{D} = \epsilon\boldsymbol{E}, \boldsymbol{B} = \mu\boldsymbol{H})$



Maxwell's Equations

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = i\omega\mu\boldsymbol{H}(\boldsymbol{r})$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = -i\omega\varepsilon\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{J}(\boldsymbol{r})$$

$$\nabla \cdot \boldsymbol{H}(\boldsymbol{r}) = 0$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \rho(\boldsymbol{r})/\varepsilon$$

Electric-Field Integral Equation

$$\boldsymbol{E}(\boldsymbol{r}) = i\omega\mu\int_V d\boldsymbol{r}'\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{J}(\boldsymbol{r}')$$

Magnetic-Field Integral Equation

$$\boldsymbol{H}(\boldsymbol{r}) = \int_V d\boldsymbol{r}'\boldsymbol{J}(\boldsymbol{r}')\times\nabla' g(\boldsymbol{r},\boldsymbol{r}')$$

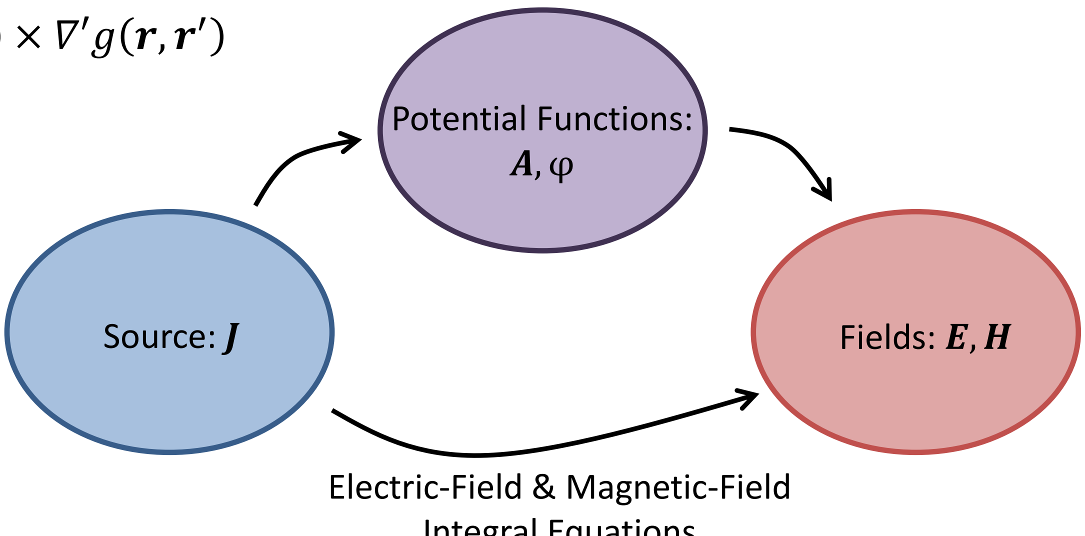
Green's Functions

Green's function of scalar Helmholtz equation in Cartesian coordinate system

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

Dyadic Green's function

$$\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}') = \left[\overline{\boldsymbol{I}} + \frac{\nabla\nabla}{k^2}\right]g(\boldsymbol{r},\boldsymbol{r}')$$



Maxwell's Equations with Magnetic Sources

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = i\omega \boldsymbol{B}(\boldsymbol{r}) - \boldsymbol{M}(\boldsymbol{r})$$ ← magnetic current density

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = -i\omega \boldsymbol{D}(\boldsymbol{r}) + \boldsymbol{J}(\boldsymbol{r})$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = \rho_m(\boldsymbol{r})$$ ← magnetic charge density

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho(\boldsymbol{r})$$

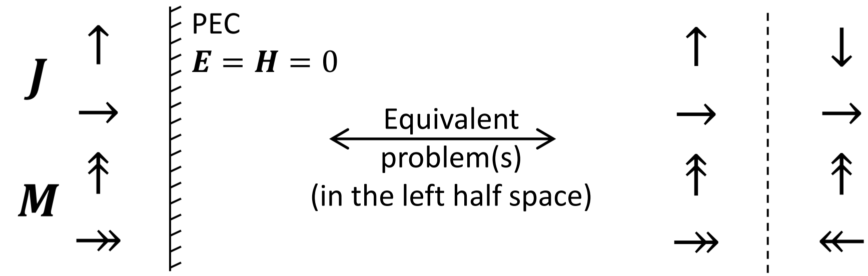
Duality:

$\boldsymbol{E} \rightarrow \boldsymbol{H}$		$\boldsymbol{M} \rightarrow -\boldsymbol{J}$	
$\boldsymbol{H} \rightarrow \boldsymbol{E}$		$\boldsymbol{J} \rightarrow -\boldsymbol{M}$	
$\boldsymbol{B} \rightarrow -\boldsymbol{B}$		$\rho_m \rightarrow -\rho$	
$\boldsymbol{D} \rightarrow -\boldsymbol{D}$		$\rho \rightarrow -\rho_m$	

Equivalence Principle

«When two source specifications give the same solution in a limited region of interest, the two problems are called equivalent.»

Image Theory

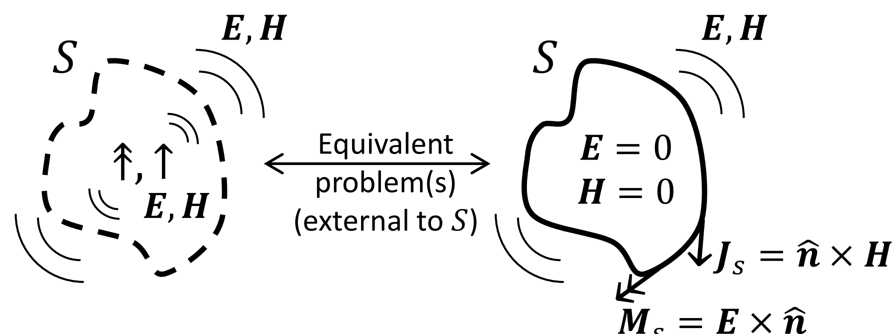


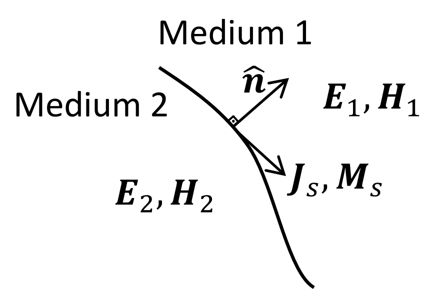
Uniqueness Conditions

The solution will be unique in a volume enclosed by a surface S if the boundary conditions are specified such that

1) tangential E field is specified over the whole surface S , or
2) tangential H field is specified over the whole surface S , or
3) tangential E field is specified over a part of S, and tangential H field is specified over the rest of S.

Huygen's Principle





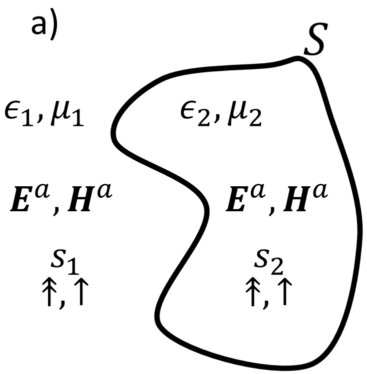
Boundary Conditions

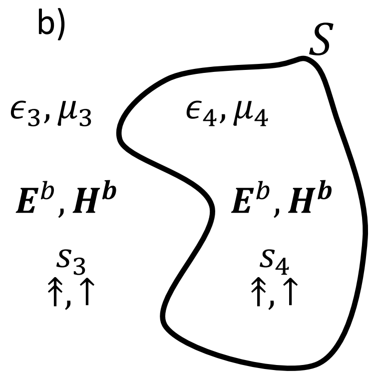
$$\hat{\boldsymbol{n}} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = -\boldsymbol{M}_s$$

$$\hat{\boldsymbol{n}} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_s$$

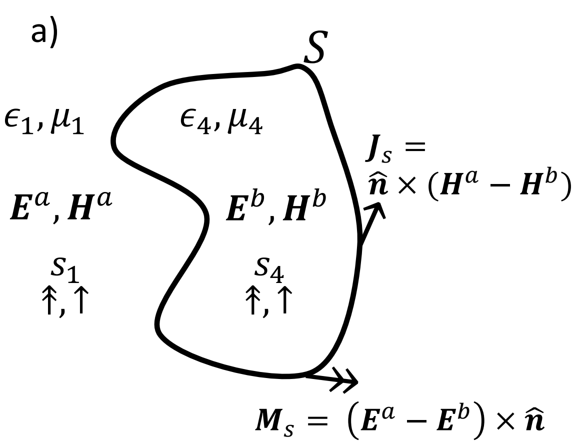
General Formulation of Surface Equivalence Principle

Two original problems:

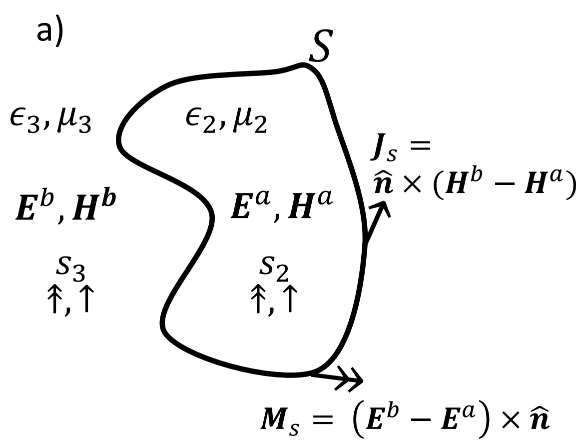




<u>Equivalent Problem:</u>
equivalent to a) external to S
equivalent to b) internal to S



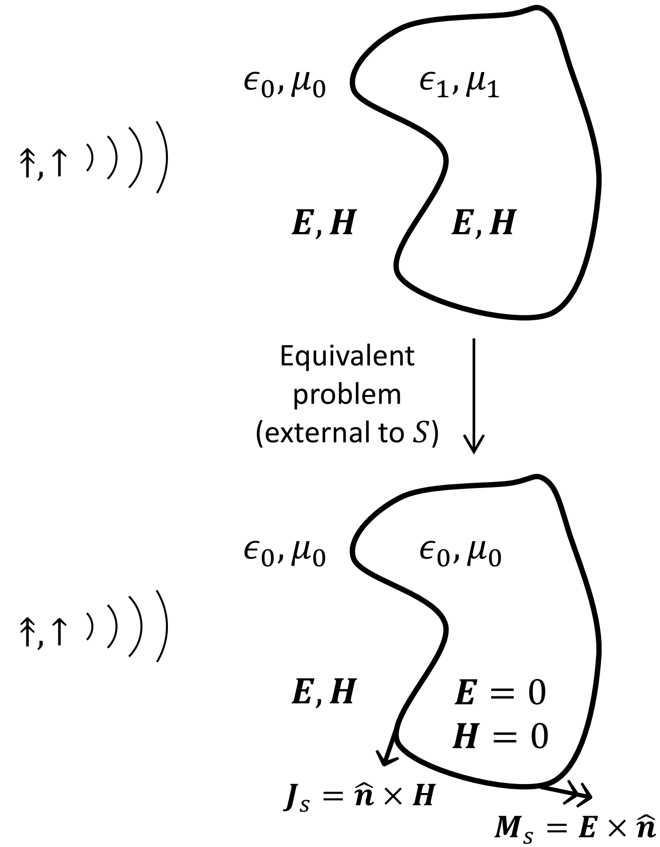
<u>Equivalent Problem:</u>
equivalent to b) external to S
equivalent to a) internal to S



Scattering From Dielectric Objects

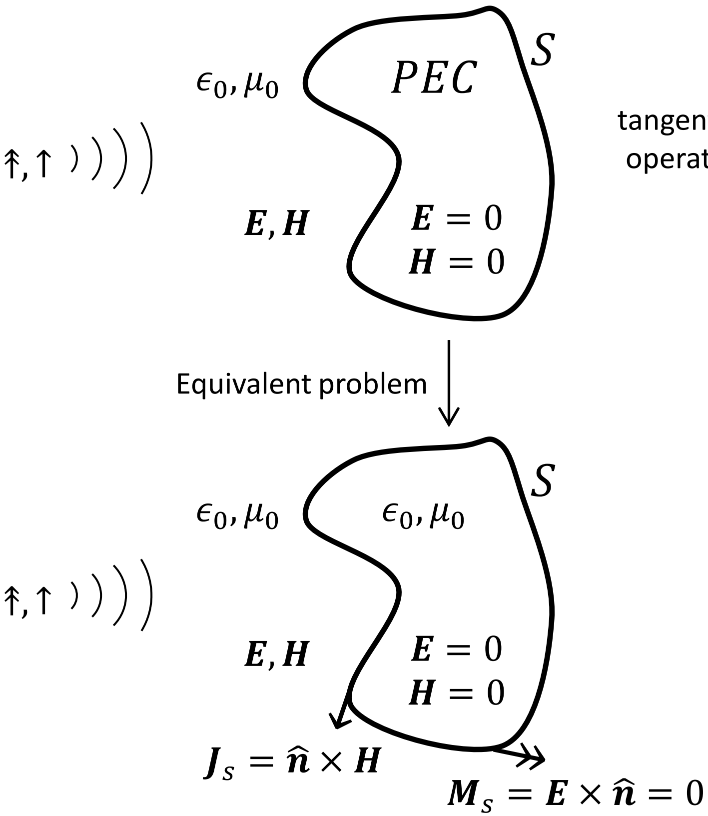
total field, incident field, scattered field

$$\boldsymbol{E} = \boldsymbol{E}^{inc} + \boldsymbol{E}^{sca}$$
$$\boldsymbol{H} = \boldsymbol{H}^{inc} + \boldsymbol{H}^{sca}$$



- Scattered fields are due to surface currents
- Fields should be Maxwellian
- $\boldsymbol{J}^s$ and $\boldsymbol{M}^s$ are the unknowns and can be solved with an integral equation

Scattering from PEC Objects



Electric-Field Integral Equation (EFIE)

tangential operator →

$$\hat{\boldsymbol{t}} \cdot \boldsymbol{E}(\boldsymbol{r}) = 0, \boldsymbol{r} \in S$$

$$\hat{\boldsymbol{t}} \cdot \left(\boldsymbol{E}^{inc}(\boldsymbol{r}) + \boldsymbol{E}^{sca}(\boldsymbol{r})\right) = 0, \boldsymbol{r} \in S$$

$$i\omega\mu\, \hat{\boldsymbol{t}} \cdot \int_S d\boldsymbol{r}' \overline{\boldsymbol{G}}(\boldsymbol{r}, \boldsymbol{r}') \cdot \underbrace{\boldsymbol{J}_s(\boldsymbol{r}')}_{\text{unknown}} = -\hat{\boldsymbol{t}} \cdot \underbrace{\boldsymbol{E}^{inc}(\boldsymbol{r})}_{\text{known}}, \boldsymbol{r} \in S$$

Magnetic-Field Integral Equation (MFIE)

$$\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{J}_s(\boldsymbol{r}), \boldsymbol{r} \in S$$

$$\hat{\boldsymbol{n}} \times \left(\boldsymbol{H}^{inc}(\boldsymbol{r}) + \boldsymbol{H}^{sca}(\boldsymbol{r})\right) = \boldsymbol{J}_s(\boldsymbol{r}), \boldsymbol{r} \in S$$

$$\underbrace{\boldsymbol{J}_s(\boldsymbol{r})}_{\text{unknown}} - \hat{\boldsymbol{n}} \times \int_S d\boldsymbol{r}' \underbrace{\boldsymbol{J}_s(\boldsymbol{r}')}_{\text{unknown}} \times \nabla' g(\boldsymbol{r}, \boldsymbol{r}') = \hat{\boldsymbol{n}} \times \underbrace{\boldsymbol{H}^{inc}(\boldsymbol{r})}_{\text{known}}, \boldsymbol{r} \in S$$

$$\boldsymbol{E} = \boldsymbol{E}^{inc} + \boldsymbol{E}^{sca}(\boldsymbol{J}_s)$$

$$\boldsymbol{H} = \boldsymbol{H}^{inc} + \boldsymbol{H}^{sca}(\boldsymbol{J}_s)$$

$$\boldsymbol{E}^{sca}(\boldsymbol{r}) = i\omega\mu \int_S d\boldsymbol{r}' \overline{\boldsymbol{G}}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{J}_s(\boldsymbol{r}')$$

$$\boldsymbol{H}^{sca}(\boldsymbol{r}) = \int_S d\boldsymbol{r}' \boldsymbol{J}_s(\boldsymbol{r}') \times \nabla' g(\boldsymbol{r}, \boldsymbol{r}')$$

Alternative representation with T, K, and I operators:

$$T\{\boldsymbol{X}\}(\boldsymbol{r}) = ik \int_S d\boldsymbol{r}' \left[\boldsymbol{X}(\boldsymbol{r}') + \frac{1}{k^2} \nabla' \cdot \boldsymbol{X}(\boldsymbol{r}') \nabla\right] g(\boldsymbol{r}, \boldsymbol{r}')$$

$$K\{\boldsymbol{X}\}(\boldsymbol{r}) = \int_S d\boldsymbol{r}' \boldsymbol{X}(\boldsymbol{r}') \times \nabla' g(\boldsymbol{r}, \boldsymbol{r}')$$

$$I\{\boldsymbol{X}\}(\boldsymbol{r}) = \boldsymbol{X}(\boldsymbol{r})$$

$$EFIE\colon\ \eta\, \hat{\boldsymbol{t}} \cdot T\{\boldsymbol{J}_s\}(\boldsymbol{r}) = -\hat{\boldsymbol{t}} \cdot \boldsymbol{E}^{inc}(\boldsymbol{r}), \boldsymbol{r} \in S$$

$$MFIE\colon \boldsymbol{J}_s(\boldsymbol{r}) - \hat{\boldsymbol{n}} \times K\{\boldsymbol{J}_s\}(\boldsymbol{r}) = \hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r}), \boldsymbol{r} \in S$$

Hypersingularity in MFIE and its Extraction

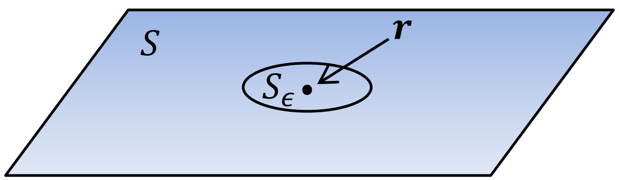
$$MFIE:\ \boldsymbol{H}^{sca}(\boldsymbol{r}) = \int_S d\boldsymbol{r}'\boldsymbol{J}_s(\boldsymbol{r}') \times \nabla' g(\boldsymbol{r},\boldsymbol{r}')$$

$g(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$: singular at $\boldsymbol{r}' = \boldsymbol{r}$

$\nabla' g(\boldsymbol{r},\boldsymbol{r}')$ has a higher-order singularity than g itself

Reminder

$$\nabla' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

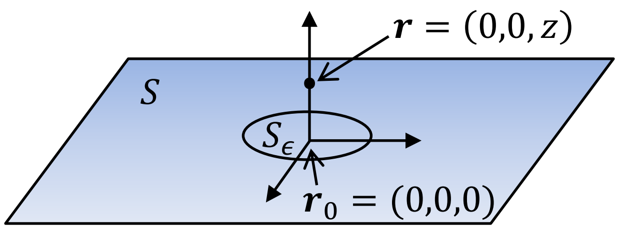


$$\boldsymbol{H}^{sca}(\boldsymbol{r}) = \underbrace{\lim_{S_\epsilon\to 0}\int_{S-S_\epsilon} d\boldsymbol{r}'\boldsymbol{J}_s(\boldsymbol{r}') \times \nabla' g(\boldsymbol{r},\boldsymbol{r}')}_{principle\ value:\ \boldsymbol{H}^{sca}_{PV}} + \underbrace{\lim_{S_\epsilon\to 0}\int_{S_\epsilon} d\boldsymbol{r}'\boldsymbol{J}_s(\boldsymbol{r}') \times \nabla' g(\boldsymbol{r},\boldsymbol{r}')}_{limit\ value:\ \boldsymbol{H}^{sca}_{lim}}$$

As $\boldsymbol{r}$ approaches S_ϵ, $g(\boldsymbol{r},\boldsymbol{r}') \cong \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$

assuming that the current is continous

$$\boldsymbol{H}^{sca}_{lim}(\boldsymbol{r}) = \lim_{\boldsymbol{r}\to\boldsymbol{r}_0}\lim_{S_\epsilon\to 0}\int_{S_\epsilon} d\boldsymbol{r}'\boldsymbol{J}_s(\boldsymbol{r}') \times \nabla' g(\boldsymbol{r},\boldsymbol{r}') = \frac{\boldsymbol{J}_s(\boldsymbol{r}_0)}{4\pi} \times \int_{S_\epsilon} d\boldsymbol{r}' \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$



$$\boldsymbol{H}^{sca}_{lim}(0,0,z) = \frac{\boldsymbol{J}_s(\boldsymbol{r}_0)}{4\pi} \times \hat{\boldsymbol{z}} \int_0^{2\pi} d\varphi' \int_0^{\epsilon} d\rho'\rho' \frac{z}{\sqrt{(\rho')^2+z^2}^3}$$

$$-\frac{\boldsymbol{J}_s(\boldsymbol{r}_0)}{4\pi} \times \hat{\boldsymbol{x}} \int_0^{2\pi} d\varphi' \int_0^{\epsilon} d\rho'(\rho')^2 \frac{\cos\varphi'}{\sqrt{\cdots}^3} \to 0 \ - \frac{\boldsymbol{J}_s(\boldsymbol{r}_0)}{4\pi} \times \hat{\boldsymbol{y}} \int_0^{2\pi} d\varphi' \int_0^{\epsilon} d\rho'(\rho')^2 \frac{\sin\varphi'}{\sqrt{\cdots}^3} \to 0$$

$$= \frac{\boldsymbol{J}_s(\boldsymbol{r}_0)}{4\pi} \times \hat{\boldsymbol{z}} \int_0^{2\pi} d\varphi' \int_0^{\epsilon} d\rho'\rho' \frac{z}{\sqrt{(\rho')^2+z^2}^3}$$

$$= \frac{\boldsymbol{J}_s(\boldsymbol{r}_0)}{2} \times \hat{\boldsymbol{z}} \left.\frac{z}{\sqrt{(\rho')^2+z^2}^3}\right]_{\rho'=0}^{\rho'=\epsilon} = \frac{\boldsymbol{J}_s(\boldsymbol{r}_0)}{2\pi} \times \hat{\boldsymbol{z}} \left(\frac{z}{\sqrt{\epsilon^2+z^2}} - \frac{z}{|z|}\right)$$

$$\boldsymbol{H}^{sca}_{lim}(0,0,z) = \lim_{z\to 0}\frac{\boldsymbol{J}_s(\boldsymbol{r}_0)}{2} \times \hat{\boldsymbol{z}} \left(\frac{z}{\sqrt{\epsilon^2+z^2}} - \frac{z}{|z|}\right) = \frac{\boldsymbol{J}_s(\boldsymbol{r}_0)}{2} \times \hat{\boldsymbol{z}}$$

$$\boldsymbol{r} = \hat{\boldsymbol{z}}z$$
$$\boldsymbol{r}' = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y = \hat{\boldsymbol{\rho}}'\rho$$

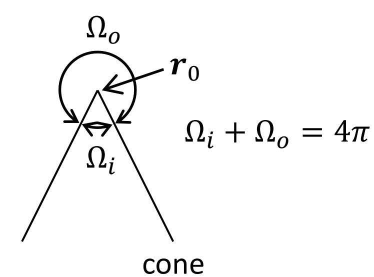
$In\ general,\ \boldsymbol{H}^{sca}_{lim}(\boldsymbol{r}_0) = \frac{\boldsymbol{J}_s(\boldsymbol{r}_0)}{2} \times \hat{\boldsymbol{n}}$ ← for smooth surfaces

$More\ generally,\ \boldsymbol{H}^{sca}_{lim}(\boldsymbol{r}_0) = \frac{\Omega_i(\boldsymbol{r}_0)}{4\pi}\boldsymbol{J}_s(\boldsymbol{r}_0) \times \hat{\boldsymbol{n}}(\boldsymbol{r}_0)$ ← solid angle

Solid Angle

$Differential: \sin\theta d\theta d\varphi$

$Total: \int_0^{2\pi}\int_0^{\pi} d\varphi \sin\theta d\theta = 4\pi$



Hypersingularity in MFIE and its Extraction

$$\boldsymbol{J}_s(\boldsymbol{r}) = \hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r}) = \hat{\boldsymbol{n}} \times \boldsymbol{H}^{sca}(\boldsymbol{r}) + \hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r})$$

$$= \hat{\boldsymbol{n}} \times \boldsymbol{H}^{sca}_{lim}(\boldsymbol{r}) + \hat{\boldsymbol{n}} \times \boldsymbol{H}^{sca}_{PV}(\boldsymbol{r}) + \hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r})$$

$$= \underbrace{\hat{\boldsymbol{n}} \times \left(\frac{\Omega_i}{4\pi}\boldsymbol{J}_s(\boldsymbol{r}) \times \hat{\boldsymbol{n}}\right)}_{\frac{\Omega_i}{4\pi}\boldsymbol{J}_s(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}) - \hat{\boldsymbol{n}}\underbrace{(\hat{\boldsymbol{n}}\cdot\boldsymbol{J}_s)}_{0} = \frac{\Omega_i}{4\pi}\boldsymbol{J}_s} + \hat{\boldsymbol{n}} \times \boldsymbol{H}^{sca}_{PV}(\boldsymbol{r}) + \hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r})$$

$$-\hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r}) = \underbrace{\left(\frac{\Omega_i}{4\pi} - 1\right)}_{\to -\frac{\Omega_o}{4\pi}}\boldsymbol{J}_s(\boldsymbol{r}) + \hat{\boldsymbol{n}} \times \boldsymbol{H}^{sca}_{PV}(\boldsymbol{r})$$

$$-\hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r}) = -\frac{\Omega_o}{4\pi}\boldsymbol{J}_s(\boldsymbol{r}) + \hat{\boldsymbol{n}} \times \boldsymbol{H}^{sca}_{PV}(\boldsymbol{r})$$

$$-\hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r}) = -\frac{\Omega_o}{4\pi}\boldsymbol{J}_s(\boldsymbol{r}) + \hat{\boldsymbol{n}} \times \int_{S_{PV}} d\boldsymbol{r}'\boldsymbol{J}_s(\boldsymbol{r}') \times \nabla' g(\boldsymbol{r},\boldsymbol{r}')$$

MFIE:

$$\boldsymbol{J}_s(\boldsymbol{r}) - \hat{\boldsymbol{n}} \times K\{\boldsymbol{J}_s\}(\boldsymbol{r}) = \hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r}), \boldsymbol{r} \in S$$

singularity extraction

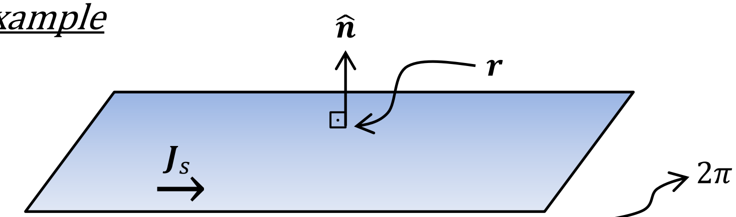
$$\frac{\Omega_o}{4\pi}\boldsymbol{J}_s(\boldsymbol{r}) - \hat{\boldsymbol{n}} \times K_{PV}\{\boldsymbol{J}_s\}(\boldsymbol{r}) = \hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r}), \boldsymbol{r} \in S$$

BAC-CAB Rule

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\mathbf{b})$$

- MFIE works on closed surfaces, not open surfaces.
- If $\boldsymbol{J}_s(\boldsymbol{r})$ is planar, and if $\boldsymbol{r}$ is on that plane, then the tangential components of $\boldsymbol{H}^{sca}_{PV}(\boldsymbol{r})$ are zero.

Example

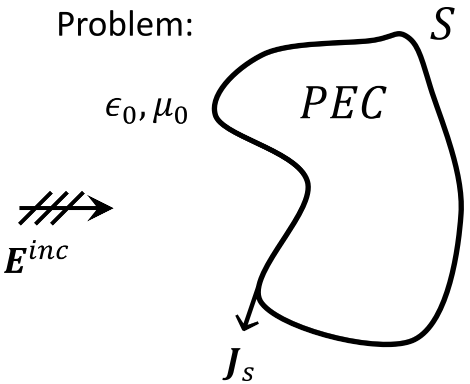


$$\hat{\boldsymbol{n}} \times \boldsymbol{H}^{sca}_{PV}(\boldsymbol{r}) = 0 \Rightarrow -\hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r}) = -\frac{\Omega_o}{4\pi}\boldsymbol{J}_s(\boldsymbol{r})$$

$$\underbrace{\boldsymbol{J}_s(\boldsymbol{r}) = 2\hat{\boldsymbol{n}} \times \boldsymbol{H}^{inc}(\boldsymbol{r})}_{Physical\ Optics}$$

Post Processing

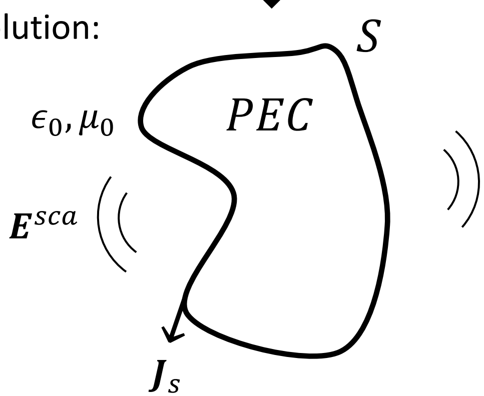
Problem:



Known: $\boldsymbol{E}^{inc}$

Unknown: $\boldsymbol{J}_s$

Solution:

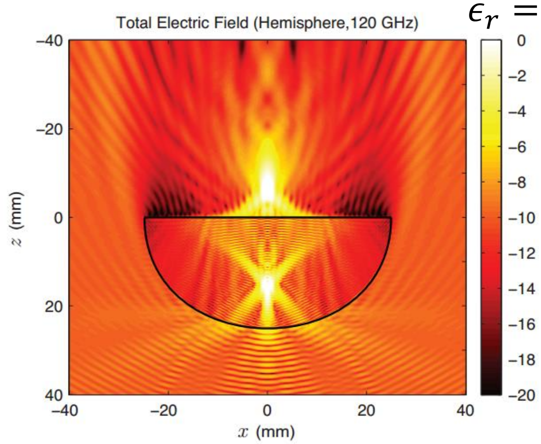


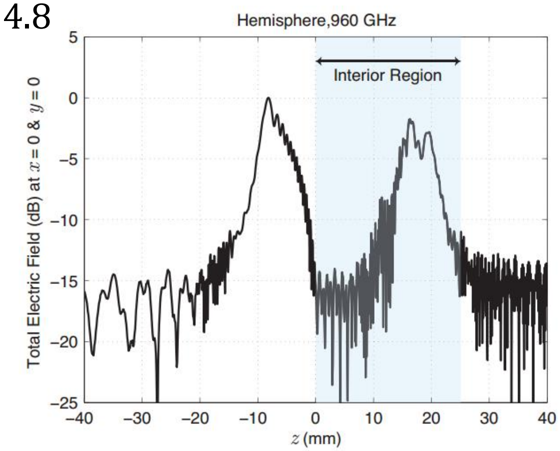
$$\boldsymbol{E} = \boldsymbol{E}^{inc} + \boldsymbol{E}^{sca}$$

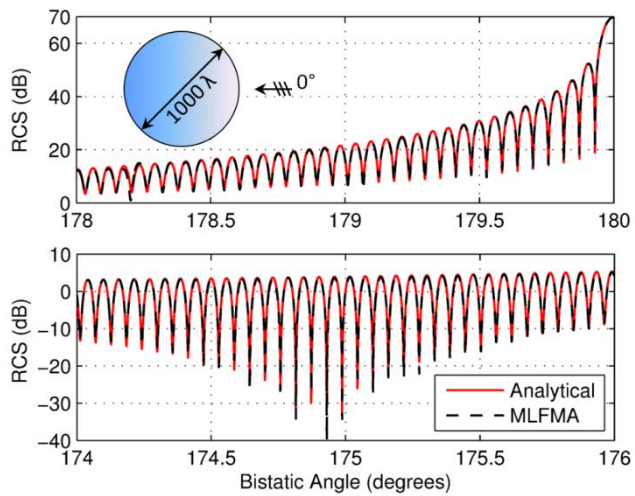
$$\boldsymbol{E}^{sca}(\boldsymbol{r}) = i\omega\mu \int_V d\boldsymbol{r}' \overline{\boldsymbol{G}}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')$$

$r = 25\ mm$

$\epsilon_r = 4.8$







Radar-Cross Section

$$E_\theta^\infty(\theta, \varphi) = \lim_{r\to\infty} re^{-ikr} E_\theta^{sca}$$

$$E_\varphi^\infty(\theta, \varphi) = \lim_{r\to\infty} re^{-ikr} E_\varphi^{sca}$$

$$RCS(\theta, \varphi) = 4\pi \left(|E_\theta^\infty(\theta, \varphi)|^2 + |E_\varphi^\infty(\theta, \varphi)|^2 \right)$$