Integral Equations in Electromagnetics

Mert Hidayetoğlu
Ultrafast Optics & Lasers Laboratory

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Electromagnetics Problem → (Method of Moments) → $\overline{\boldsymbol{A}}_{N\times N}\cdot\boldsymbol{x}=\boldsymbol{b}$

unknown: $\boldsymbol{x}$

known: $\overline{\boldsymbol{A}}_{N\times N}$, $\boldsymbol{b}$

Memory Complexity:

$$O(N^2)+O(N)+O(N)=O(N^2)$$

Computational Complexity:

Gaussian Elimination, LU Decomposition: $O(N^3)$

Iterative Solution: $O(N^2)$

Iterative Solver

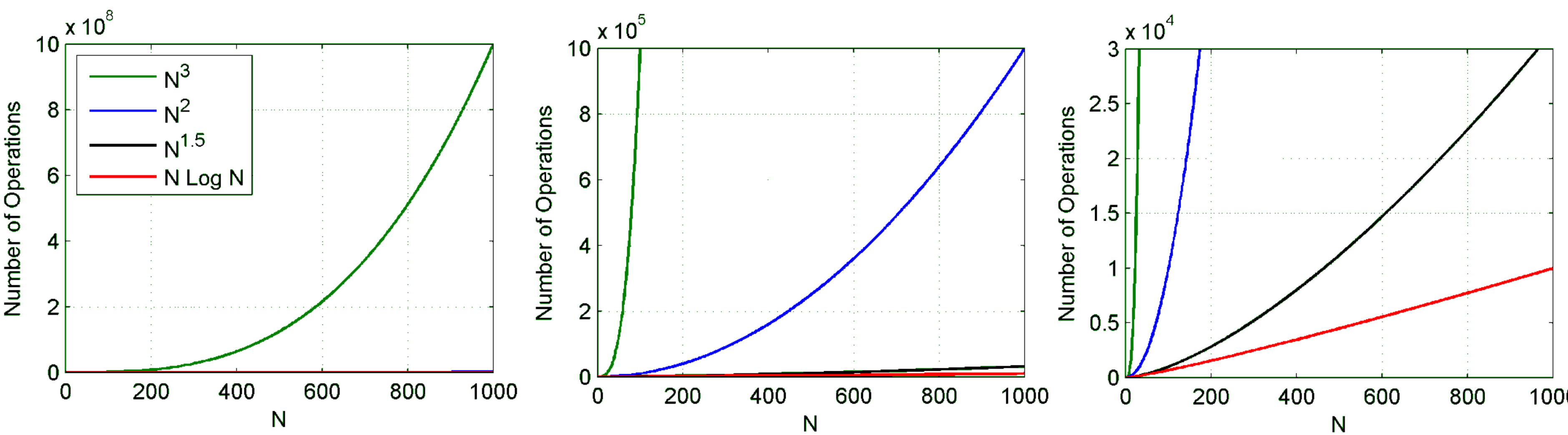
Initial Guess: $\boldsymbol{x}_0$

Substitute: $\overline{\boldsymbol{A}}\cdot\boldsymbol{x}-\boldsymbol{b}=\boldsymbol{r}_0$

$$\boldsymbol{r}_{i-1}\rightarrow\boxed{Solver}\rightarrow\boldsymbol{x}_i\rightarrow\boxed{\overline{\boldsymbol{A}}\cdot\boldsymbol{x}}-\boldsymbol{b}\rightarrow\boldsymbol{r}_i$$

$O(N^2)$

Fast solvers: $O(N^3)\rightarrow O(N^2)\rightarrow O(N^{1.5})\rightarrow O(N\log N)$

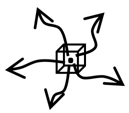


Maxwell's Equations

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{d}{dt}\boldsymbol{B}(\boldsymbol{r},t)$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \frac{d}{dt}\boldsymbol{D}(\boldsymbol{r},t) + \boldsymbol{J}(\boldsymbol{r},t)$$

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r},t) = -\frac{d}{dt}\rho(\boldsymbol{r},t)$$ ⟵ Continuity Equation



decreasing charge at a point
creates a current divergence

$$\nabla \cdot (\nabla \times \boldsymbol{E}) = -\frac{d}{dt}\nabla \cdot \boldsymbol{B} = 0 \Rightarrow \nabla \cdot \boldsymbol{B} = 0,$$

$$\nabla \cdot (\nabla \times \boldsymbol{H}) = \nabla \cdot \left(\boldsymbol{J} + \frac{d\boldsymbol{D}}{dt}\right) = 0$$

$$\frac{d\nabla \cdot \boldsymbol{D}}{dt} = -\nabla \cdot \boldsymbol{J} = \frac{d}{dt}\rho \Rightarrow \nabla \cdot \boldsymbol{D} = \rho,$$

$$where\ \boldsymbol{D}(\boldsymbol{r},t=0) = \rho(\boldsymbol{r},t=0) = \boldsymbol{B}(\boldsymbol{r},t=0) = 0$$

$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{d}{dt}\boldsymbol{B}(\boldsymbol{r},t)$ ⟵ Faraday's Law

$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \frac{d}{dt}\boldsymbol{D}(\boldsymbol{r},t)$ ⟵ Ampere's Law

$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0$ ⟵ Magnetic Flux Law

$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho(\boldsymbol{r},t)$ ⟵ Gauss' Law

$\boldsymbol{E}$: *electric field intensity* $\left(\frac{V}{m}\right)$
$\boldsymbol{H}$: *magnetic field intensity* $\left(\frac{A}{m}\right)$
$\boldsymbol{B}$: *magnetic flux density* $\left(\frac{Wb}{m^2}\right)$
$\boldsymbol{D}$: *electric flux density* $\left(\frac{C}{m^2}\right)$
$\boldsymbol{J}$: *Current density* $\left(\frac{A}{m^2}\right)$
ρ: *charge density* $\left(\frac{C}{m^3}\right)$

Vector Identity:

$\nabla \cdot (\nabla \times \boldsymbol{A}) \equiv 0$

«The divergence of the curl of any vector field is zero»

Continuity Equation from Maxwell's Eq.

$\nabla \cdot (\nabla \times \boldsymbol{H}) = \nabla \cdot \left(\boldsymbol{J} + \frac{d}{dt}\boldsymbol{D}\right) = 0$

$\Rightarrow \nabla \cdot \boldsymbol{J} = -\frac{d}{dt}\nabla \cdot \boldsymbol{D} = -\frac{d}{dt}\rho$

Time-Harmonic Maxwell's Equations

Single-frequency sinsoidal time variation: $\boldsymbol{E}(\boldsymbol{r},t) = Re\{\boldsymbol{E}(\boldsymbol{r})e^{-i\omega t}\}$

$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = i\omega \boldsymbol{B}(\boldsymbol{r})$ ⟵ Faraday's Law

$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = -i\omega \boldsymbol{D}(\boldsymbol{r}) + \boldsymbol{J}(\boldsymbol{r})$ ⟵ Ampere's Law

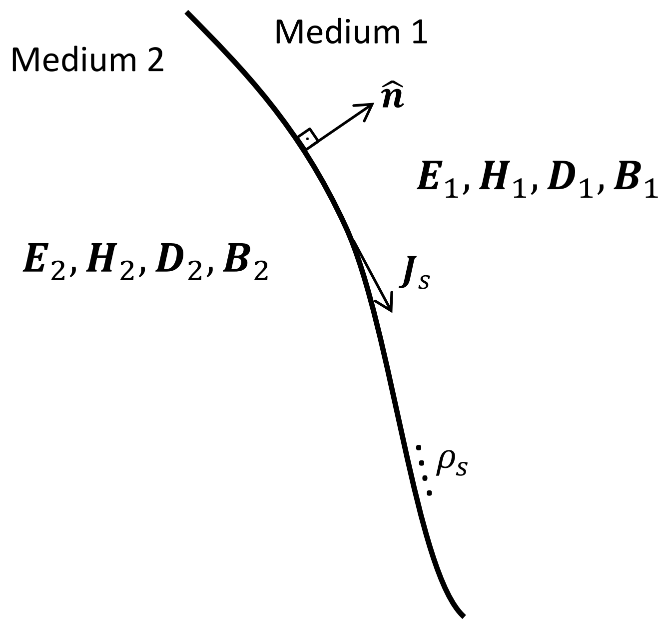
$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0$ ⟵ Magnetic Flux Law

$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho(\boldsymbol{r})$ ⟵ Gauss' Law

$\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = i\omega\rho(\boldsymbol{r})$ ⟵ Continuity Equation

Time-derivative of Phasors

$$\frac{d}{dt}\boldsymbol{E}(\boldsymbol{r},t) = Re\{\underbrace{-i\omega \boldsymbol{E}(\boldsymbol{r})}_{\text{phasor}}e^{-i\omega t}\}$$



Boundary Conditions

$$\hat{\boldsymbol{n}} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0$$
$$\hat{\boldsymbol{n}} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_s$$
$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0$$
$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \rho_s$$

Sources Creating Fields: Potential Functions

Objective: Find the fields due to $\boldsymbol{J}(\boldsymbol{r})$ in a homogenious and isotropic medium.

electric permittivity → ϵ; magnetic permeability → μ

$$(\boldsymbol{D} = \epsilon\boldsymbol{E}, \boldsymbol{B} = \mu\boldsymbol{H})$$

$$\nabla \times \boldsymbol{E} = i\omega\mu\boldsymbol{H}$$
$$\nabla \times \boldsymbol{H} = -i\omega\epsilon\boldsymbol{E} + \boldsymbol{J}$$

$$\nabla \cdot (\mu\boldsymbol{H}) = 0 \Rightarrow \mu\boldsymbol{H} = \nabla \times \boldsymbol{A}$$

(vector magnetic potential: $\boldsymbol{A}$)

$$\nabla \times \boldsymbol{E} = i\omega\nabla \times \boldsymbol{A} \Rightarrow \nabla \times (\boldsymbol{E} - i\omega\boldsymbol{A}) = 0$$
$$\Rightarrow \boldsymbol{E} - i\omega\boldsymbol{A} = -\nabla\varphi$$
$$\Rightarrow \boldsymbol{E} = i\omega\boldsymbol{A} - \nabla\varphi$$

(scalar electric potential: φ)

$$\mu\nabla \times \boldsymbol{H} = \nabla \times (\nabla \times \boldsymbol{A})$$
$$\mu(-i\omega\epsilon\boldsymbol{E} + \boldsymbol{J}) = \nabla \cdot (\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A}$$
$$-i\omega\mu\epsilon(i\omega\boldsymbol{A} - \nabla\varphi) + \mu\boldsymbol{J} = \nabla \cdot (\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A}$$
$$\nabla^2\boldsymbol{A} + \omega^2\mu\epsilon\boldsymbol{A} - \nabla(\underbrace{\nabla \cdot \boldsymbol{A}} - i\omega\mu\epsilon\varphi) = -\mu\boldsymbol{J}$$

Lorentz Gauge: $\nabla \cdot \boldsymbol{A} = i\omega\mu\epsilon\varphi$

(*Coulomb Gauge in static case*: $\nabla \cdot \boldsymbol{A} = 0$)

$$\nabla^2\boldsymbol{A} + \omega^2\mu\epsilon\boldsymbol{A} = -\mu\boldsymbol{J}$$

Identities

I) $\nabla \times \nabla\varphi \equiv 0$

II) $\nabla \cdot (\nabla \times \boldsymbol{A}) \equiv 0$

Helmholtz Theorem

«A vector field is determined if both its divergence and its curl are specified everywhere.»

Reminder

∇ Del operator: $\nabla\varphi, \nabla \cdot \boldsymbol{A}, \nabla \times \boldsymbol{A}$

∇^2 Laplacian operator: $\nabla^2\varphi = \nabla \cdot \nabla\varphi$

∇^2 Vector Laplacian operator: $\nabla^2\boldsymbol{A} = \nabla \cdot (\nabla \cdot \boldsymbol{A}) - \nabla \times (\nabla \times \boldsymbol{A})$

$$\underbrace{\nabla \cdot \boldsymbol{E}}_{\rho/\epsilon} = i\omega\underbrace{\nabla \cdot \boldsymbol{A}}_{i\omega\mu\epsilon\varphi} - \underbrace{\nabla \cdot \nabla}_{\nabla^2}\varphi$$

$$\nabla^2\varphi + \omega^2\mu\epsilon\varphi = -\frac{\rho}{\epsilon}$$

$$\left.\begin{aligned} \nabla^2 A_x + k^2 A_x &= -\mu J_x \\ \nabla^2 A_y + k^2 A_y &= -\mu J_y \\ \nabla^2 A_z + k^2 A_z &= -\mu J_z \\ \nabla^2\varphi + k^2\varphi &= -\frac{1}{\epsilon}\rho \end{aligned}\right\}$$

Non – Homogeneous Helmholtz's Equations

wavenumber: $k = \omega\sqrt{\mu\epsilon}$

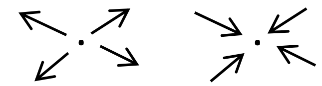
Green's Function of Helmholtz Equation

Objective: Solve $\nabla^2\psi(\boldsymbol{r}) + k^2\psi(\boldsymbol{r}) = -\delta(\boldsymbol{r}-\boldsymbol{r}')$ *everywhere except at* $\boldsymbol{r}'$.

$$\boldsymbol{r}' = \boldsymbol{0} \Rightarrow \psi(\boldsymbol{r}) = \psi(x,y,z) = \psi(r,\theta,\varphi) = \psi(r) : spherical\ symmetry\ around\ origin.$$

$$\nabla^2\psi = \frac{1}{r}\frac{d^2}{dr^2}(r\psi) \Rightarrow \frac{d^2}{dr^2}(r\psi) + k^2(r\psi) = 0 \Rightarrow general\ solution:\ \psi = \frac{c_1}{r}\underbrace{e^{ikr}}_{\text{outgoing wave}} + \frac{c_2}{r}\underbrace{e^{-ikr}}_{\text{incoming wave}}$$

$$Physically:\ c_2 = 0 \Rightarrow \psi = \frac{c_1}{r}e^{ikr}$$



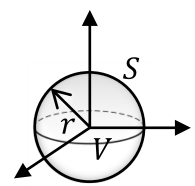
Particular solution involving a point source at the origin: substitute ψ into the non-homogeneous Helmhotz Equation and integrate boths sides:

$$\int_V \nabla^2\psi(\boldsymbol{r})d\boldsymbol{r} + k^2\int_V \psi(\boldsymbol{r})d\boldsymbol{r} = -\int_V \delta(\boldsymbol{r})d\boldsymbol{r}$$

$$\oint_S \nabla\psi\cdot d\boldsymbol{S} + 4\pi k^2\int_0^r r'^2\psi dr' = -1$$

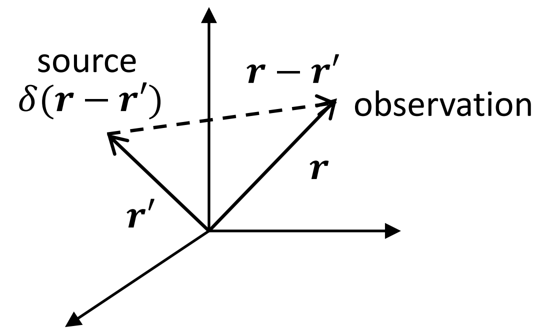
$$-4\pi r^2 c_1\left(\frac{e^{ikr}}{r^2} - ik\frac{e^{ikr}}{r}\right) + 4\pi k^2 c_1\left(\frac{re^{ikr}}{ik} + \frac{e^{ikr}-1}{k^2}\right) = -1$$

$$r \to 0 \Rightarrow c_1 = \frac{1}{4\pi}$$



Gauss' Divergence Law:

$$\int_V \nabla\cdot\boldsymbol{A}dV = \oint_S \boldsymbol{A}\cdot d\boldsymbol{S}$$



$$Solution:\ c_1 = \frac{1}{4\pi} \Rightarrow \psi(r) = \frac{e^{ikr}}{4\pi r} \Rightarrow \psi(\boldsymbol{r}) = \frac{e^{ik|\boldsymbol{r}|}}{4\pi|\boldsymbol{r}|}$$

$$For\ any\ point\ source\ located\ at\ \boldsymbol{r}', \psi(\boldsymbol{r}) = \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = g(\boldsymbol{r},\boldsymbol{r}') \longleftarrow \text{Green's function}$$

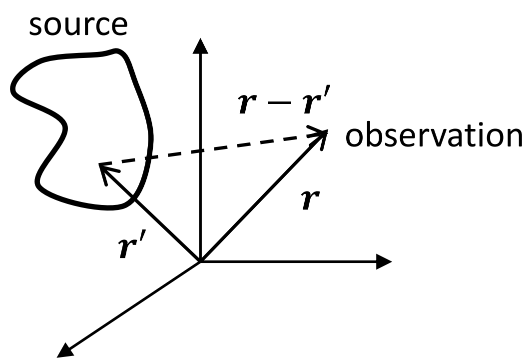
$$\nabla^2\varphi + k^2\varphi = -\frac{1}{\epsilon}\rho\ \ Helmholtz's\ Equation\ for\ Scalar\ Electric\ Potential$$

$$\rho(\boldsymbol{r}) = \rho_0\delta(\boldsymbol{r}-\boldsymbol{r}_0) \Rightarrow \varphi(\boldsymbol{r}) = \frac{1}{\epsilon}\rho_0\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}_0|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|} = \frac{1}{\epsilon}\rho_0 g(\boldsymbol{r},\boldsymbol{r}_0) = \frac{1}{\epsilon}\int_V \underbrace{\rho_0\delta(\boldsymbol{r}'-\boldsymbol{r}_0)}_{\rho(\boldsymbol{r})}g(\boldsymbol{r},\boldsymbol{r}')d\boldsymbol{r}'$$

Volume Integral Equations

$$\boldsymbol{A}(\boldsymbol{r}) = \mu \int_V d\boldsymbol{r}'\boldsymbol{J}(\boldsymbol{r}')\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$\varphi(\boldsymbol{r}) = \frac{1}{\epsilon} \int_V d\boldsymbol{r}'\rho(\boldsymbol{r}')\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$



<u>Continuity Equation:</u>

$\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = i\omega\rho(\boldsymbol{r})$

<u>Space Derivative:</u>

$\nabla \cdot (g\boldsymbol{J}) = \nabla g \cdot \boldsymbol{J} + g\nabla \cdot \boldsymbol{J}$

<u>Gauss' Divergence Law:</u>

$$\int_V \nabla \cdot (g\boldsymbol{J}) = \oint_S g\boldsymbol{J}$$

<u>Identity:</u>

$\nabla' g(\boldsymbol{r},\boldsymbol{r}') = -\nabla g(\boldsymbol{r},\boldsymbol{r}')$

$$\boldsymbol{E}(\boldsymbol{r}) = i\omega\boldsymbol{A}(\boldsymbol{r}) - \nabla\varphi(\boldsymbol{r}) = i\omega\mu \int_V d\boldsymbol{r}' g(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}') - \frac{\nabla}{\epsilon}\int_V d\boldsymbol{r}' g(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')$$

$$= i\omega\mu \int_V d\boldsymbol{r}' g(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}') - \frac{\nabla}{i\omega\epsilon}\int_V d\boldsymbol{r}' g(\boldsymbol{r},\boldsymbol{r}')\nabla' \cdot \boldsymbol{J}(\boldsymbol{r}')$$

$$\int_V d\boldsymbol{r}' g(\boldsymbol{r},\boldsymbol{r}')\nabla' \cdot \boldsymbol{J}(\boldsymbol{r}') = \int_V d\boldsymbol{r}'\nabla \cdot g(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}') - \int_V d\boldsymbol{r}'\nabla' g(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')$$

$$= \underbrace{\oint_S d\boldsymbol{S} \cdot g(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')}_{0} + \int_V d\boldsymbol{r}'\nabla g(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')$$

$$= i\omega\mu \int_V d\boldsymbol{r}' g(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}') + \frac{\nabla}{i\omega\epsilon}\int_V d\boldsymbol{r}'\nabla g(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')$$

<u>Unit Dyad:</u>

$$\bar{\boldsymbol{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Outer Product:</u>

$$\boldsymbol{ab} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix}$$

unit dyad — outer product — dyadic Green's function

$$\boldsymbol{E}(\boldsymbol{r}) = i\omega\mu\left[\bar{\boldsymbol{I}} + \frac{\nabla\nabla}{k^2}\right] \cdot \int_V d\boldsymbol{r}' g(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}') = i\omega\mu \int_V d\boldsymbol{r}'\bar{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')$$

<u>Dyadic Green's Function:</u>

$$\bar{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}') = \left[\bar{\boldsymbol{I}} + \frac{\nabla\nabla}{k^2}\right] g(\boldsymbol{r},\boldsymbol{r}')$$

$$\boldsymbol{H}(\boldsymbol{r}) = \frac{1}{\mu}\nabla \times \boldsymbol{A}(\boldsymbol{r}) = \nabla \times \int_V d\boldsymbol{r}' g(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}') = \int_V d\boldsymbol{r}'\nabla g(\boldsymbol{r},\boldsymbol{r}') \times \boldsymbol{J}(\boldsymbol{r}') = \int_V d\boldsymbol{r}'\boldsymbol{J}(\boldsymbol{r}') \times \nabla' g(\boldsymbol{r},\boldsymbol{r}')$$

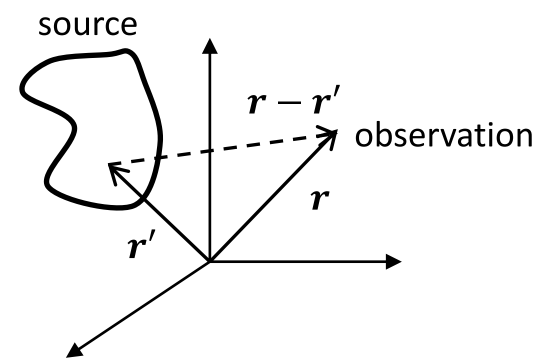
From Sources to Fields: Integral Equations

Electric − Field Integral Equation

$$\boldsymbol{E}(\boldsymbol{r}) = i\omega\mu \int_V d\boldsymbol{r}'\,\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')$$

Magnetic − Field Integral Equation

$$\boldsymbol{H}(\boldsymbol{r}) = \int_V d\boldsymbol{r}'\,\boldsymbol{J}(\boldsymbol{r}') \times \nabla' g(\boldsymbol{r},\boldsymbol{r}')$$



<u>Green's Functions</u>

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}') = \left[\overline{\boldsymbol{I}} + \frac{\nabla\nabla}{k^2}\right] g(\boldsymbol{r},\boldsymbol{r}')$$