INTERVAL CONSTRAINED EQUILIBRIA IN BIPOLAR AND TRIPOLAR SYSTEMS

A. Bülent Özgüler
Electrical and Electronics Engineering Department

Serdar Ş. Güner*
Department of International Relations

Nedim M. Alemdar
Department of Economics

Bilkent University
Bilkent, Ankara
06533 Turkey
phone: 90-312-2902070/2901249
fax: 90-312-2664326
email: sguner@bilkent.edu.tr

1 MAIN RESULTS

Given a set of states $\mathcal{N} = \{1, ..., n\}$ with $n \geq 2$, suppose that for all $i \in \mathcal{N}$, the i-th state has a resource $r_i \geq 0$ some portion of which it uses to allocate against the remaining states $\mathcal{N} - \{i\}$. Let a_{ij} denote the allocation of state-i against the state-j, with a_{ii} denoting the self allocation or internal consumption by state-i. Suppose that each a_{ij} satisfies

$$a_{ij} \in [\alpha_{ij}, \beta_{ij}] \ \ \forall \ \ i, j \in \mathcal{N}, \tag{1}$$

where $\alpha_{ij} \leq \beta_{ij}$ are real numbers specified such that

$$[\alpha_{ij}, \beta_{ij}] \subset [0, r_i] \ \ \forall \ \ i, j \in \mathcal{N}. \tag{2}$$

For $i \neq j$, the interval $[\alpha_{ij}, \beta_{ij}]$ can be viewed as the *threat perception by state-i of state-j*. If state-i perceives j as a big threat against its security, then α_{ij} would be chosen to be close to r_i by state-i; otherwise, if j appears to pose a small threat to state-i, then β_{ij} would be chosen to be close to zero. For $i = j$, the interval $[\alpha_{ii}, \beta_{ii}]$ can be viewed as the *vehemence attributed by state-i to internal security*, alternatively, the lust of state-i for internal consumption.

Remark 1. An alternative interpretation for the interval $[\alpha_{ij}, \beta_{ij}]$ is as follows. Suppose the defensive and the offensive capabilities of a state are not necessarily the same. It is somewhat reasonable to assume that the minimum defensive capability is more than the maximum offensive capability. Let α_{ij} represent *the maximum possible offensive allocation* and β_{ij} *the minimum possible defensive allocation of i against j* for all $\{i, j\} \subset \mathcal{N}$. The interval $[\alpha_{ij}, \beta_{ij}]$ then represents *the interval of comfort for j*, since any allocation choice $a_{ij} \in [\alpha_{ij}, \beta_{ij}]$ by i is neither enough to beat j nor enough to defend against j. The interval $[\alpha_{ii}, \beta_{ii}]$ is again the interval of self-allocation of state-i for $i \in \mathcal{N}$. $\triangle$

Since the whole r_i is either used for internal consumption or allocated against the remaining states, a_{ij}'s satisfy

$$\sum_{j \in \mathcal{N}} a_{ij} = r_i \ \ \forall \ \ i \in \mathcal{N}. \tag{3}$$

We call the world of n states to be *in interval constrained equilibrium*, or *ic-equilibrium* for short, if in every one-against-one confrontation the allocated resources of the two states are the same, i.e., if (1), (3) hold, and

$$a_{ij} = a_{ji} \ \ \forall \ \ \{i, j\} \subset \mathcal{N}. \tag{4}$$

We are mainly concerned with the question:

Q1. *Given n states with resources $r_1, ..., r_n$ and intervals $[\alpha_{ij}, \beta_{ij}]$ $\forall i, j \in \mathcal{N}$ obeying (2), is it possible to choose allocations a_{ij} $\forall$ $i, j \in \mathcal{N}$ such that the world is in ic-equilibrium?*

The world will be called *ic-balanced* if the answer to (Q1) is in the affirmative, i.e., if there exist allocations $a_{ij}\ \forall\ i,j \in \mathcal{N}$ such that the world is in ic-equilibrium.

The answer tightly depends on the resource values as well as the specified intervals. If for instance $\alpha_{ij} = \beta_{ij} = 0$, or on the other extreme $\alpha_{ij} = \beta_{ij} = r_i$, for all $i,j \in \mathcal{N}$, then an ic-equlibrium exists only in the trivial case of all states having no resource. If the intervals for $i \neq j$ are chosen to be the largest possible and if no internal consumption is allowed, i.e.,

$$\begin{array}{l} [\alpha_{ij}, \beta_{ij}] = [0, r_i]\ \ \forall\ \ \{i,j\} \subset \mathcal{N}, \\ \alpha_{ii} = \beta_{ii} = 0\ \ \forall\ \ i \in \mathcal{N}, \end{array} \tag{5}$$

then ic-equilibrium is possible if and only if no state controls more than half of the total world resource by a result of Özgüler, Güner, and Alemdar, 1998. We state this result for later reference within the framework of ic-balance:

Fact 1. *A world of n states with resources r_i satisfying $r_i > 0\ \ \forall\ \ i \in \mathcal{N}$ and intervals satisfying (5) is ic-balanced if and only if*

$$r_i \leq \sum_{j \in \mathcal{N} - \{i\}} r_j\ \ \ \forall\ \ i \in \mathcal{N}. \tag{6}$$

Thus, under the restrictions (5) for intervals, the existence of an ic-equilibrium depends on whether the resource values satisfy (6) or not. The condition (6) is known as the *no hegemony condition.* If all intervals are chosen to be at their largest, i.e., if

$$[\alpha_{ij}, \beta_{ij}] = [0, r_i]\ \ \forall\ \ i,j \in \mathcal{N}, \tag{7}$$

then, whatever the resource values are, there is always an ic-equilibrium. In fact, if the condition (6) is satisfied so that there is no hegemon, then an ic-equilibrium can be attained upon zero internal consumptions by the states. If there is a hegemon, say state-1, then $r_1 - \sum_{j=2}^{n} r_j > 0$. Letting $a_{11} = r_1 - \sum_{j=2}^{n} r_j$, $a_{ij} = 0; i,j = 2, ..., n$, and $a_{i1} = a_{1i} = r_i$, an ic-equilibrium is achieved. We thus have the following result.

Fact 2. *A world of n states with resources r_i satisfying $r_i > 0\ \ \forall\ \ i \in \mathcal{N}$ and intervals satisfying (7) is ic-balanced.*

The examples above show that the intervals must be consistent in order for an ic-equilibrium to exist. Even in case where only (3) is required, the intervals should be consistent as the following shows.

Fact 3. *Given $r_i \geq 0\ \ \forall\ \ i \in \mathcal{N}$ and intervals satisfying (5), there exist $a_{ij} \geq 0\ \ \forall\ \ i,j \in \mathcal{N}$ satisfying (1) and (3) if and only if*

$$\sum_{j \in \mathcal{N}} \alpha_{ij} \leq r_i \leq \sum_{j \in \mathcal{N}} \beta_{ij}\ \ \ \ \forall\ \ i \in \mathcal{N}. \tag{8}$$

Proof. If $a_{ij} \geq 0$ satisfies (1), then

$$\alpha_{ij} \leq a_{ij} \leq \beta_{ij} \;\; \forall \;\; i,j \in \mathcal{N}.$$

Summation of each term for $j = 1, ..., n$ and (3) gives $\alpha_i \leq r_i \leq \beta_i$ for all $i \in \mathcal{N}$, where

$$\alpha_i := \sum_{j \in \mathcal{N}} \alpha_{ij}, \quad \beta_i := \sum_{j \in \mathcal{N}} \beta_{ij}.$$

This yields (8). Conversely, If (8) holds, then let

$$a_{ij} := \frac{r_i}{2}(\frac{\alpha_{ij}}{\alpha_i} + \frac{\beta_{ij}}{\beta_i}) \;\; \forall \;\; i,j \in \mathcal{N}.$$

It follows that (1) and (3) hold. □

The condition (8) together with $\alpha_{ij} \leq \beta_{ij}$ is equivalent to

$$max\,\{\alpha_{ij}, r_i - \sum_{k \in \mathcal{N}-\{j\}} \beta_{ik}\} \leq min\,\{\beta_{ij}, r_i - \sum_{k \in \mathcal{N}-\{j\}} \alpha_{ik}\} \;\; \forall \;\; i,j \in \mathcal{N}. \tag{9}$$

This inequality has a simple intuitive interpretation. Both arguments of *max* in (9) are *the minimum possible allocations of state-i against state-j*. Similarly, both arguments of *min* in (9) are *the maximum possible allocations of state-i against state-j*. The condition (9), or the condition (8), reads: *the minimum possible allocations of i against j should not exceed its maximum possible allocations.* We will refer to (9) as the *consistency condition for intervals.*

The following intermediate result shows that, the general problem of the existence of an ic-equilibrium is equivalent to the solvability of $n(n+1)$ inequalities in $n(n-1)/2$ unknowns. In (11) below, the first and the second sum is zero for $i = 1$ and $i = n$, respectively.

Lemma 1. *Given resources $r_i \geq 0 \;\; \forall \;\; i \in \mathcal{N}$ and intervals $[\alpha_{ij}, \beta_{ij}] \;\; \forall i,j \in \mathcal{N}$ obeying (2) there exist $a_{ij} \geq 0 \;\; \forall \;\; i,j \in \mathcal{N}$ satisfying (1), (3), and (4) if and only if there exist $x_{ij} \;\; \forall \;\; i < j;\; i,j \in \mathcal{N}$ satisfying*

$$max\,\{\alpha_{ij}, \alpha_{ji}\} \leq x_{ij} \leq min\,\{\beta_{ij}, \beta_{ji}\} \;\; \forall \;\; i < j;\; i,j \in \mathcal{N} \tag{10}$$

and

$$r_i - \beta_{ii} \leq \sum_{k=1}^{i-1} x_{ki} + \sum_{l=i+1}^{n} x_{il} \leq r_i - \alpha_{ii} \;\; \forall \;\; i \in \mathcal{N}. \tag{11}$$

Proof. Suppose there exist a_{ij}'s satisfying (1), (3), and (4). Let $x_{ij} := a_{ij} = a_{ji}$ for all $i < j$. By (1), the inequalities (10) are satisfied. Moreover, by (3), it follows that

$$a_{ii} + \sum_{k=1}^{i-1} x_{ki} + \sum_{l=i+1}^{n} x_{il} = r_i \;\; \forall \;\; i \in \mathcal{N}$$

which, using (1) for $i = j$, yields (11). Conversely, if (10) and (11) hold for some x_{ij}'s, then let $a_{ij} := x_{ij}$ and $a_{ji} := x_{ij}$ for all $i < j$; $i, j \in \mathcal{N}$. It follows that (4) is satisfied. Also let

$$a_{ii} := r_i - \sum_{k=1}^{i-1} x_{ki} - \sum_{l=i+1}^{n} x_{il}.$$

The requirement (3) is also satisfied. Finally, by (10) and (11), the condition (1) holds.□

1.1 A Bipolar World

We first give an answer to (Q1) for a two-country world, i.e., for $n = 2$.

The main result is that *a bipolar world is ic-balanced, i.e., an ic-equilibrium exists, if and only if the minimum possible allocations of state-i against state-j does not exceed the maximum possible allocations of state-j against state-i for* $i, j \in \{1, 2\}$.

Fact 4. *A bipolar world of resources* r_1, r_2 *and intervals* $[\alpha_{ij}, \beta_{ij}]$, $i, j \in \{1, 2\}$ *is ic-balanced if and only if*

$$\begin{array}{l} max\,\{\alpha_{12}, r_1 - \beta_{11}\} \leq min\,\{\beta_{12}, r_1 - \alpha_{11}\}, \\ max\,\{\alpha_{21}, r_2 - \beta_{22}\} \leq min\,\{\beta_{21}, r_2 - \alpha_{22}\} \end{array} \tag{12}$$

and

$$\begin{array}{l} max\,\{\alpha_{12}, r_1 - \beta_{11}\} \leq min\,\{\beta_{21}, r_2 - \alpha_{22}\}, \\ max\,\{\alpha_{21}, r_2 - \beta_{22}\} \leq min\,\{\beta_{12}, r_1 - \alpha_{11}\}. \end{array} \tag{13}$$

Proof. By Lemma 1 with $n = 2$, an ic-equilibrium exists if and only if there exists $x = x_{12}$ satisfying

$$\begin{array}{l} max\,\{\alpha_{12}, \alpha_{21}\} \leq x \leq min\,\{\beta_{12}, \beta_{21}\}, \\ r_1 - \beta_{11} \leq x \leq r_1 - \alpha_{11}, \\ r_2 - \beta_{22} \leq x \leq r_2 - \alpha_{22}. \end{array}$$

These inequalities are equivalent to

$$max\,\{\alpha_{12}, \alpha_{21}, r_1 - \beta_{11}, r_2 - \beta_{22}\} \leq x \leq min\,\{\beta_{12}, \beta_{21}, r_1 - \alpha_{11}, r_2 - \alpha_{22}\}. \tag{14}$$

Now if (14) holds, then the consistency condition (12) as well as the condition (13) hold. Conversely if (12) and (13) hold, then

$$max\,\{\alpha_{12}, \alpha_{21}, r_1 - \beta_{11}, r_2 - \beta_{22}\} \leq min\,\{\beta_{12}, \beta_{21}, r_1 - \alpha_{11}, r_2 - \alpha_{22}\}$$

and there exists x satisfying (14). $\square$

The condition (12) is the consistency condition (9) for the intervals and is known to be necessary (for any n) for an ic-equilibrium, by Fact 3. The additional condition (13) has a similar interpretation to the consistency condition, namely, the minimum possible allocations of i against j should not exceed the maximum possible allocations of j against i, for $i, j \in \{1, 2\}$ and $i \neq j$. The combined condition can be phrased as in the italicized statement given at the beginning of this subsection.

1.2 A Tripolar World

Let us now consider the case of three countries, i.e., $n = 3$. The foregoing analysis of a bipolar world suggests that the condition for ic-balance will involve minimum and maximum possible mutual allocations of states. In the tripolar world, the determination of such extreme allocations require more care. If we consider the minimum possible allocation of state-i against state-j, for instance, in a tripolar world each of the following quantities represents a minimal possible allocation by state-i against state-j:

α_{ij}	*lower limit of the threat interval*
$r_i - \beta_{ii} - \beta_{ik}$	*left-over resource after maximal self allocation and maximal allocation to k*
$r_i - \beta_{ii} - \beta_{ki}$	*left-over resource after maximal self allocation and counteracting a maximal allocation by k*
$r_i - \beta_{ii} - (r_k - \alpha_{kk} - \alpha_{kj})$	*left-over resource after maximal self allocation and counteracting a maximal allocation by k*
$r_i - \beta_{ii} - (r_k - \alpha_{kk} - \alpha_{jk})$	*left-over resource after maximal self allocation and counteracting a maximal allocation by k*

The above list can be actually be continued *ad infinitum* by substituting alternative expressions for α_{jk}, α_{kj} in the last two quantities and by substituting alternative expressions for β_{ji}, β_{ij} in the new quantities obtained. It will turn out however that the following definitions of minimum and maximum possible allocations by states suffice to characterize ic-balance in a tripolar world:

$$\begin{aligned} miAj \;&:=\; max\{\; \alpha_{ij}, r_i - \beta_{ii} - \beta_{ik}, r_i - \beta_{ii} - \beta_{ki}, \\ &\qquad\quad\;\; r_i - \beta_{ii} - (r_k - \alpha_{kk} - \alpha_{kj}), r_i - \beta_{ii} - (r_k - \alpha_{kk} - \alpha_{jk})\}, \\ MjAi \;&:=\; min\{\; \beta_{ji}, r_j - \alpha_{jj} - \alpha_{jk}, r_j - \alpha_{jj} - \alpha_{kj}, \\ &\qquad\quad\;\; r_j - \alpha_{jj} - (r_k - \beta_{kk} - \beta_{ki}), r_j - \alpha_{jj} - (r_k - \beta_{kk} - \beta_{ik})\}. \end{aligned} \tag{15}$$

Fact 5. *A tripolar world of resources $r_i,\ i \in \{1,2,3\}$ and intervals $[\alpha_{ij}, \beta_{ij}],\ i,j \in \{1,2,3\}$ is ic-balanced if and only if (9) holds and*

$$miAj \leq MjAi \ \ \forall \ \ \{i,j\} \subset \{1,2,3\}. \tag{16}$$

A proof of Fact 5 is given in the Appendix.

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