1. Let $\mathcal{D}$ and $\tilde{\mathcal{D}}$ be two DSR’s having the same input space.

   (a) Show that if $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are equivalent, then they have the same input-output pairs. By an input-output pair, we mean a pair $(u_{[t_0,\infty)}, y_{[t_0,\infty)})$ of waveforms, where $t_0 \in \mathcal{T}$ and $y_{[t_0,\infty)}$ is the output resulting from the input $u_{[t_0,\infty)}$ and some initial state $x_0 \in \Sigma$.

   (b) Show that the following finite-state automata have the same input-output pairs. Are they equivalent?

2. Following is a typical input/output pair for a device. Can you associate a dynamical system representation with this device? If yes, identify the relevant constituents; if no, explain.

3. Let $\mathcal{D} = (\mathcal{T}, \mathcal{U}, \Sigma, \mathcal{Y}, s, r)$ and $\tilde{\mathcal{D}} = (\mathcal{T}, \mathcal{U}, \Sigma, \mathcal{Y}, s, \tilde{r})$ be two DSR’s where

   \[ r(t, x(t), u(t)) = x(t) \]

   Show that
(a) if $D$ is linear and $\forall t \in T, \bar{r}(t, \ldots) : \Sigma \times U \rightarrow Y$ is a linear map, then $\bar{D}$ is also linear,
(b) if $D$ is time-invariant and $\bar{r}(t, x(t), u(t)) = \bar{r}(x(t), u(t))$ (that is, $\bar{r}$ does not depend explicitly on $t$), then $\bar{D}$ is also time-invariant.

4. A discrete-time system is described by
$$x(k+1) = x(k) + \min \{ u(k) - x(k), 0 \}$$
where $u$ is the input and $x$ is the output. Associate a dynamical system representation with this system. Is your DSR linear? Is it time-invariant?

5. Repeat Q.4 for the continuous-time system
$$\dot{x}(t) = \min \{ u(t) - x(t), 0 \}$$

6. Recall that the state transition function of the dynamical system representation associated with an ideal delay was defined as
$$s(t_1, t_0, x_{t_0}, u) = \begin{cases} (P_{t_1}u)_{[-\tau, 0)}, & t_1 \geq t_0 + \tau \\ (P_{t_1-t_0}x_{t_0})_{[-\tau, t_0-t_1]} \ast (P_{t_1}u)_{[t_0-t_1, 0)}, & t_1 < t_0 + \tau \end{cases}$$
Show that $s$ satisfies the semigroup axiom. [Hint: Consider the cases $t_2 \geq t_1 + \tau; t_2 < t_1 + \tau, t_1 \geq t_0 + \tau; \text{and} \ t_2 < t_1 + \tau, t_1 < t_0 + \tau$ separately.]

7. Consider the following system, where S & H stands for a sample-and-hold device whose input/output characteristics are defined as
$$v(t) = u(kT), kT \leq t < (k+1)T; k \in \mathbb{Z}; t, T \in \mathbb{R}$$
a) Associate a dynamical system representation $D$ with this system.
8. Consider a SISO system

\[ \dot{x}(t) = Ax(t) + bu(t) \]
\[ y(t) = c^T x(t) + du(t) \]

Suppose that the input of the system is generated by sampling an external signal periodically and holding the sampled values until next sampling so that \( u(t) = v_k, \quad kT \leq t < (k + 1)T \). Letting \( x_k = x(kT), y_k = y(kT) \) we define a discrete-time system which relates input sequences \((v_k)\) to output sequences \((y_k)\). Obtain a representation of this system.