1. (a) Prove that the pair \((A, B)\), where \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\) is controllable if and only if the matrix

\[
\begin{bmatrix}
B & I & 0 & 0 & \cdots & 0 \\
-A & B & I & 0 & \cdots & 0 \\
-A & B & I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-A & B & I & 0 & \cdots & 0 \\
-A & B & I & 0 & \cdots & 0 \\
\end{bmatrix}
\]

has rank \(n^2\).

(b) State a similar result for observability of a pair \((C, A)\).

2. Consider a feedback connection of two systems \(S_i : \dot{x}_i = A_i x_i + B_i u_i, y_i = C_i x_i, i = 1, 2\), where \(S_1\) is in the forward path and \(S_2\) is in the feedback path.

(a) Show that the feedback connection is controllable if and only if the cascade connection in which \(S_1\) is followed by \(S_2\) is controllable.

(b) Show that the feedback connection is observable if and only if the cascade connection in which \(S_2\) is followed by \(S_1\) is observable.

3. The controllability Grammian \(W\) of a stable, controllable system \((A, B)\) is the unique symmetric positive-definite solution of \(AW + WA^T = -BB^T\). Let \(W\) have the singular-value decomposition \(W = V \Sigma^2 V^T\), let \(T = V \Sigma\), and define \(\tilde{A} = T^{-1} AT, \tilde{B} = T^{-1} B\). Find the controllability Grammian of the system \((\tilde{A}, \tilde{B})\).

4. Consider the system

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]

Find a piecewise constant input of the form

\[
u(t) = \begin{cases} \alpha, & 0 \leq t < 1 \\ \beta, & 1 \leq t < 2 \end{cases}
\]

which will transfer the system from the initial state \(x(0) = [0 \ 2]^T\) to the final state \(x(2) = [4 \ 0]^T\).
5. Obtain Kalman’s canonical decomposition of the system

\[ S: \dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} u \]

\[ y = [3 \quad -1 \quad 1] x \]

and obtain the representation of a controllable and observable system which is zero-state equivalent to \( S \).

6. Find Kalman’s canonical decomposition of

\[ A = \begin{bmatrix} -3 & 2 & 2 \\ -1 & 0 & 2 \\ 1 \quad -1 \quad -3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad c^T = [-1 \quad 2 \quad 2]^T \]

and obtain a minimal system having the same transfer function.

7. Let \( S_1(A_1, B_1, C_1) \) and \( S_2(A_2, B_2, C_2) \) be two minimal realizations of order \( n \) of a given transfer function matrix \( H(s) \). Show that \( A_2 = T^{-1}A_1T, B_2 = T^{-1}B_1, C_2 = C_1T \) for some nonsingular \( T \), that is \( S_1 \) and \( S_2 \) are algebraically equivalent. \textbf{Hint:} Follow the steps below.

\begin{enumerate}
  \item \( M_1, M_2, N_1, N_2 \) all have rank \( n \).
  \item \( M_1M_1^T, M_2M_2^T, N_1^T N_1, N_2^T N_2 \) are all nonsingular.
  \item With \( T = (N_2^T N_2)^{-1}N_2^T N_1, T^{-1} = M_1M_2^T (M_2M_2^T)^{-1}. \)
\end{enumerate}

8. Obtain minimal realizations of

\[ H_1(s) = \begin{bmatrix} \frac{2}{s+2} & \frac{s}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}, \quad H_2(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{s+1} \\ 0 & \frac{1}{s+3} \end{bmatrix}, \quad H_3(s) = \begin{bmatrix} s^{-1} & 0 \\ s^{-1} & s^{-2} \end{bmatrix} \]

9. Consider the discrete–time system in Q.28, where \( \mu \) is known but the initial state \( x(0) = [\alpha \beta]^T \) is unknown. Construct a \textbf{finite} input sequence \( \{u(k)\} \) based on the information \( \{y(0), y(1), ..., y(k-1)\} \) available at time \( k \), such that \( y(k) = 0 \) for \( k \geq K \) with \( K \) minimum.

10. Given a scalar transfer function \( h(s) = (s + 1)/s^3 \).

\begin{enumerate}
  \item Obtain the observable canonical realization \((c^T, A, b)\) of \( h(s) \).
  \item Find a nonsingular transformation matrix \( T \) such that \((c^T T, T^{-1}AT, T^{-1}b)\) is the controllable canonical realization of \( h(s) \).
\end{enumerate}
11. Find a state-space description of a minimal system which has the same transfer function as

\[
\begin{align*}
\dot{x} &= \begin{bmatrix}
\lambda & 1 \\
\lambda & 1 \\
\mu & 1 \\
\mu & 1
\end{bmatrix} x + \begin{bmatrix}
1 \\
0 \\
-2 \\
1
\end{bmatrix} \text{un} \times n \\
y &= \begin{bmatrix}
1 & 2 & 0 & 1
\end{bmatrix} \text{xn} \times n
\end{align*}
\] (5)

12. Given a linear, time-invariant, continuous-time system

\[
\dot{x} = \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix} x + \begin{bmatrix}
1 \\
0
\end{bmatrix} u
\]

a) Find the state-transition matrix.

b) Check controllability of the system at \( t_0 = 0 \). If it is not completely controllable, characterize controllable states.

c) Find a piecewise constant input of the form

\[
u(t) = \begin{cases}
\alpha, & 0 \leq t < 1 \\
\beta, & 1 \leq t < 2
\end{cases}
\]

which transfers the system from the initial state \( x(0) = \begin{bmatrix} 0 & 2 \end{bmatrix}^T \) to the final state \( x(2) = \begin{bmatrix} 2 & 0 \end{bmatrix}^T \).

13. Consider a continuous-time system represented by

\[
\begin{align*}
\dot{x} &= \begin{bmatrix}
-\sigma & \omega \\
-\omega & -\sigma
\end{bmatrix} x + \begin{bmatrix}
0 \\
1
\end{bmatrix} \text{un} \times n \\
y &= \begin{bmatrix}
1 & 0
\end{bmatrix} \text{xn} \times n
\end{align*}
\] (7)

a) Calculate \( \Phi(t, t_0) \).

b) Is the system completely controllable? Completely observable?

c) Suppose that the input of the system is generated from a sequence \( u^d(k) \) as \( u(t) = u^d(k), kT \leq t < (k+1)T \), and the output is sampled to obtain a sequence \( y^d(k) = y(kT) \). Define \( x^d(k) = x(kT) \) and obtain a state-space description of the resulting discrete-time system with input \( u^d \) and output \( y^d \).

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d) Determine under what conditions on the sampling period $T$ the discrete-time system is completely controllable and completely observable.

14. a) Show that a pair $(J, B)$, where $J$ is in Jordan form, is controllable if and only if the rows of $B$ corresponding to the last rows of the Jordan subblocks associated with the same eigenvalue are linearly independent.

b) State a similar result for observability of a pair $(C, J)$.

15. Obtain the observability indices and an observable canonical form of the pair $(A, C)$, where

$$A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \hspace{1cm} C = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}$$

16. Consider a scalar transfer function

$$h(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \cdots + \frac{c_n}{s-p_n}$$

where $p_i$ are distinct. Show that $(A, B, C)$ with

$$A = \begin{bmatrix}
p_1 & & & \\
& p_2 & & \\
& & \ddots & \\
& & & p_n
\end{bmatrix} \hspace{1cm} B = \begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}$$

$$C = \begin{bmatrix}
c_1 & c_2 & \cdots & c_n
\end{bmatrix}$$

is a minimal realization of $h(s)$.

17. Obtain the observability indices and an observable canonical form of the pair $(A, C)$, where

$$A = \begin{bmatrix}
-1 & 0 & 1 \\
-1 & -2 & 0 \\
-1 & -1 & 0
\end{bmatrix} \hspace{1cm} C = \begin{bmatrix}
1 & 0 & -1 \\
1 & 1 & 0
\end{bmatrix}$$
18. Consider a discrete–time system

\[
x(k + 1) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \quad \text{for } k = 0, 1, \ldots, n - 1
\]

\[
y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k) \quad \text{for } k = 0, 1, \ldots, n - 1
\]

Suppose that at time \( k \), \( y(0), y(1), \ldots, y(k) \) and \( u(0), u(1), \ldots, u(k-1) \) are available, and that \( x(0) \) is unknown. Find a finite input sequence \( \{u(k)\} \) such that \( y(k) = (1/2)^k \) for \( k \geq K \) with minimum \( K \).

19. Consider a single-input discrete-time system \( S : x(k + 1) = Ax(k) + bu(k) \), where

\[
A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -d_n \\ 1 & 0 & \cdots & 0 & -d_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -d_1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

a) Show that \( S \) is completely reachable.

b) Find an input sequence \( u(0), u(1), \ldots, u(n - 1) \) such that \( x(n) = s(n, 0, \theta, u) = [\alpha_{n-1} \ldots \alpha_1 \alpha_0]^T \) for arbitrarily specified \( \alpha_i, i = 1, 2, \ldots, n - 1 \).

20. A discrete-time system \( S = (C, A, B) \) is said to be strongly observable if \( x(0) \) can be determined from the knowledge of a finite output sequence \( y(0), y(1), \ldots, y(K) \) without any need to know the input sequence \( u(0), u(1), \ldots, u(K - 1) \). Show that \( S \) is strongly observable if \( CB = CAB = \cdots = CA^{k_o-2}B = 0 \), where \( k_o \) is the observability index of \( S \).

21. Find an input of the form \( u(t) = \alpha + \beta t \) which would steer the system

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = u
\]

from the initial state \( x(0) = [1 \ 0]^T \) to the final state \( x(1) = [0 \ 1]^T \).

22. Find the Kalman’s canonical decomposition of the system in Q.2, and obtain the representation of a minimal system having the same transfer function.
23. Consider the system \([A, b, c^T]\) with
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
\[
c^T = \begin{bmatrix} 1 & 1 \end{bmatrix}
\]
a) Let \(x(0) = \theta\). Find a finite input of the form
\[
u(t) = \begin{cases} 
\alpha, & 0 \leq t < 1 \\
\beta, & 1 \leq t < 2 \\
0, & 2 \leq t
\end{cases}
\]
such that \(y(t) = t\) for \(t \geq 2\).
b) Can you find a finite input \(u_{[0,t_f]}\) such that \(y(t) = e^t\) for \(t \geq t_f\)? Explain.

24. (a) Find a minimal realization of
\[
\hat{h}(s) = \frac{s^2 + 1}{s^3}
\]
(b) Transform the system \([A, b, c^T]\) with
\[
A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]
\[
c^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}
\]
into controllable canonical form.