EE 501 : LINEAR SYSTEM THEORY
EXERCISES ON DESIGN

1. Given a system

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u n \times n \\
y &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x n \times n
\end{align*}
\]

(a) Design a state feedback to have closed loop eigenvalues all at \( s = -1 \).
(b) Design a full order observer with poles all at \( s = -2 \).
(c) Obtain a representation, transfer function, and the poles of the resulting closed-loop system when the feedback in part (b) is taken from the observed rather than actual states.
(d) Repeat (b) and (c) with a reduced order observer.
(e) Design a minimal order dynamic output feedback compensator of the form

\[
\begin{align*}
\dot{z} &= Fz + Gyn \times n \\
u &= Hz + Kyn \times n
\end{align*}
\]

so that 3 poles of the resulting closed-loop system are at \( s = -1 \) and the remaining at \( s = -2 \).

2. Given a system

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u n \times n \\
y &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x n \times n
\end{align*}
\]

(a) Design a state feedback to have closed loop eigenvalues all at \( s = -1 \).
   i. Design a full order observer with poles all at \( s = -2 \).
      ii. Obtain a representation, transfer function and the poles of
          the resulting closed-loop system when the feedback in part (a) is taken from the observed rather than actual states.
c) Repeat (c) with a reduced order observer.

d) Design a minimal order dynamic output feedback compensator of the form

\[
\dot{z} = Fz + Gyn \times n \\
u = Hz + Kn \times n
\]

so that 3 poles of the resulting closed-loop system are at \( s = -1 \) and the remaining at \( s = -2 \).

3. Given a system

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\
0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\
0 & 1 \\
1 & 0 \end{bmatrix} un \times n \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} xn \times n
\end{align*}
\]

a) Design a reduced-order observer and a feedback from the observed states to have closed loop eigenvalues all at \( s = -1 \).

b) Repeat (a) with a minimal order dynamic output feedback compensator.

4. In the following feedback configuration \( h(s) = 1/(s - 1) \), and \( \eta(t) = \eta_0 + \eta_1 t + \cdots + \eta_{k-1} t^{k-1} \) represents a polynomial type disturbance. Design the feedback compensator \( f(s) \) such that the closed-system is stable and with \( r = 0, y_{ss}(t) = 0 \) despite the disturbance.

5. Consider the system

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\
0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x
\end{align*}
\]

(a) Design an observer followed by feedback from the observed states

\[
\begin{align*}
\dot{x}_e &= Ax_e + bu + g(c^T x_e - y) \\
u &= k^T x_e + v
\end{align*}
\]

such that the closed-loop system has all eigenvalues at \( s = -1 \).
(b) Design a first–order dynamic output feedback controller of the form
\[
\dot{z} = pz + qy \\
u = z + fy + v
\]
such that the closed–loop system has all eigenvalues at \( s = -1 \).

(c) In each case above find the closed–loop transfer function from the external input \( v(s) \) to \( y(s) \).

1. Consider a feedback system \( \dot{x} = u, \ u = -kx \), where \( k > 0 \).

   (a) Find an expression for
   \[
   J(k, x_o) = \int_0^\infty [x^2(t) + u^2(t)]dt
   \]
   in terms of \( k \) and \( x(0) = x_o \), where \( x(t) \) is the solution of the closed–loop system.

   (b) Find \( k = k^* \) which minimizes \( J(k, x_o) \) for a fixed \( x_o \), and also find \( J^*(x_o) = J(k^*, x_o) \).

2. Consider the system
\[
\dot{x} = Ax + bu + dv \\
y = c^T x
\]
with
\[
A = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
\[
c^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]
where \( u \) is the control input, and \( v \) is a disturbance. Find a feedback control \( u(t) = f^T x(t) \) such that the closed–loop system is asymptotically stable, and the response \( \rho(t, 0, x_o, v) \) to disturbance is independent of \( v \).

3. Consider a linear, time–invariant system \( \dot{x} = Ax + Bu, y = Cx \) with
\[
A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix}
\]
(a) Obtain a modal matrix and the Jordan form of $A$.
(b) Obtain the state transition matrix $Φ(t, τ)$.
(c) Find $ρ(t, 0, x_0, θ_u)$ for $x_0 = [−2\ 1\ 1]^T$.
(d) Transform the system into controllable canonical form.
(e) Find the transfer function of the system.
(f) Obtain the steady–state response to $u(t) = t$, $t ≥ 0$.
(g) Design a state feedback to have closed–loop poles at $s_{1,2} = −1 ± j$, $s_3 = −5$.
(h) Design a reduced–order observer with poles all at $s = −5$.

4. Given a scalar discrete-time system

$$x(k + 1) = x(k) + u(k) , \ y(k) = x(k − 1)$$

Design an observer of the form

$$z(k + 1) = az(k) + bu(k) + cy(k) , \ w(k) = z(k) + dy(k)$$

such that $x(k) = w(k)$ for all $k ≥ K$ independent of $x(−1), x(0), w(0)$ with $K$ minimum possible. Hint: Define $e(k) = x(k) − w(k)$. 

32