THE DISCRETE FRACTIONAL COSINE TRANSFORM

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ABSTRACT

There is a close relationship between the conventional Discrete Cosine Transform (DCT) and Discrete Fourier Transform (DFT). Here, we introduce another transform, the Discrete Fractional Cosine Transform (DFrCT), which has a similar relationship with the Discrete Fractional Fourier Transform (DFrFT). The DFrCT share many useful properties of the regular cosine transform, and has a free parameter, its fraction. When the fraction is zero, we get the cosine modulated version of the input signal. When it is unity, we get the conventional DCT. As the fraction changes from 0 to 1 we get different forms of the signal which interpolate between the cosine modulated form of the signal and its DCT representation. Thus, DFrCT is a general form of DCT which has an additional free parameter, and with this free parameter it may find its place in many applications where DCT is found to be useful.

1. INTRODUCTION

The a^{th} order fractional Fourier transform is the generalization of the conventional Fourier transform such that a=1 corresponds to the conventional Fourier transform and a=0 corresponds to the identity operation. However, the optical and digital implementations of this more general and flexible transform is just as efficient as the conventional Fourier transform. In addition to that, it has a very close relationship with the Wigner Distribution which makes the fractional Fourier transform attractive in time-frequency analysis of signals. Specifically, the horizontal axis of the Wigner plane corresponds to the signal domain (a=0), the vertical axis corresponds to the Fourier domain (a=1), and any axis in between corresponds to the fractional Fourier domain. The effect of the fractional Fourier transform is best exploited when the signals to be transformed are concentrated neither in signal nor in Fourier domain. More about fractional Fourier transform can be found in [1-5] and its applications can be found in [6-10] and the references therin.

The fractional Fourier transform was defined in continuous domain and there was not any discrete definition of the transform. All digital implementations were based on approximating the continuous version of the definition (the reader may refer to [6] as an example). The approximations were more successful when the number of samples were larger, and they became poorer when the number of samples were fewer. Recently proposed discrete definition of the transform in [10] allowed us to define the fractional Fourier transform for any number of samples. Here we will use this discrete definition to define our discrete fractional cosine tranform (DFrCT) for any number of samples. Our DFrCT definition shares many useful properties of the regular DCT. In this paper, we will first describe the derivation of DFrCT, and then will focus on the properties of it through some illustrative examples.

2. THE DFrCT

The definition of the discrete cosine transform is:

$$v(k) = \alpha(k)\sum_{n=0}^{N-1} x(n)\cos\left(\frac{\pi(2n+1)}{2N}\right) \quad , \quad 0 \le k \le N-1$$

where

$x(n)$: input signal, $v(n)$: output signal

$$\alpha(0) = \sqrt{\frac{1}{N}}, \quad \alpha(k) = \sqrt{\frac{2}{N}} \text{ for } 1 \le k \le N-1$$

An alternative definition is in terms of the Fourier transformation of the extended signal to twice the signal size by flipping around the original signal:

$$y[n] = x[n] + x[2N - n - 1]$$

Let us define $C_x(k)$ as

$$C_x(k) = e^{-i2\pi k/4N} Y(k)$$

where Y(k) is the 2N point discrete Fourier transform of the extended signal y[n]. Then the DCT of x[n] is the first N elements of $C_x(k)$. This definition can be rewritten as F_{2N} (2N point discrete Fourier transform) and F_{2N}^{-1} (2N point inverse discrete Fourier transform) of the input x[n] as

$$C_x(k) = \cos\left(\frac{\pi k}{2N}\right)\left[F_{2N}\{x(n)\}[k] + F_{2N}^{-1}\{x(n)\}[k]\right] - i * \sin\left(\frac{\pi k}{2N}\right)\left[F_{2N}\{x(n)\}[k] - F_{2N}^{-1}\{x(n)\}[k]\right]$$

From this interpretation, we can see that the overall transformation can be rewriten in terms of matrix multiplications:

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F = 2Nx2N DFT matrix,
IF= 2Nx2N IDFT matrix,
COS_MAT = diag(cos(n * Pi / 2N))
SIN_MAT = diag(sin(n * Pi / 2N))
DCT_EXT = COS_MAT*(F + IF)
          -i*SIN_MAT*(F - IF)
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and the DCT matrix is the first NxN part of the DCT_EXT matrix.This definition is a very convenient format for defining the DFrCT. In other words, by replacing the 2Nx2N forward and inverse DFT matrices with the corresponding Discrete Fractional Fourier Transform (DFrFT) matrices defined in [10], we readily obtain the real valued DFrCT matrix.

3. PROPERTIES OF DFrCT

By construction, the DFrCT matrix at fraction = 1 is equal to the regular DCT matrix.

Due to the construction method, the DFrCT matrix shares many of the useful properties of the regular DCT matrix. Consider the following frame images of the DFrCT at various fractions. In Fig.1, we can see the DFrCT frames at fraction = 1. Due to the spectral properties of this transform, it can be observed that the spatial frequencies increase along horizontal and vertical directions. However, the transformation does not preserve the localization information, and the transform signal only contains spectral information. In many image coding algorithms, in order to exploit the different characteristics of the image (due to being non-ergodic), the transforms are performed over smaller blocks in the image [11,12].

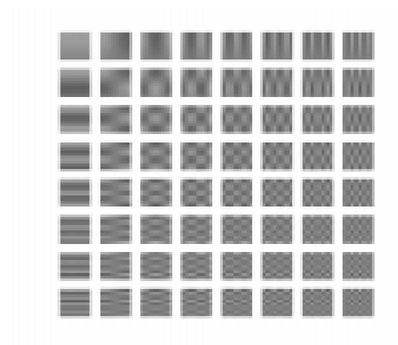

Figure 1

In Figures 2 to 6, we show DFrCT frames corresponding to fractions 0.8, 0.6, 0.4, 0.2, and 0.1, respectively. These figures illustrate an important property of the DFrCT. As we proceed from higher fractions to lower fractions, we start obtaining more localized regions on the frame images. Finally, the frame image at fraction = 0 is simply an array of matrices each of which has only one non-zero element at the raster-scan order. This situation shows that we had obtain an adjustable time localization information by using the fractional Fourier domain at different fractions.

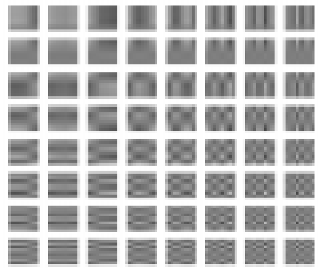

Figure 2

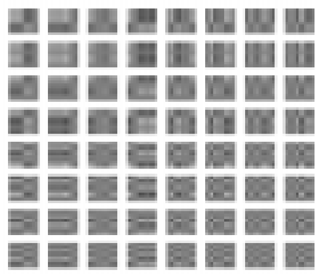

Figure 3

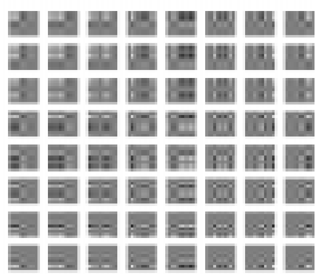

Figure 4

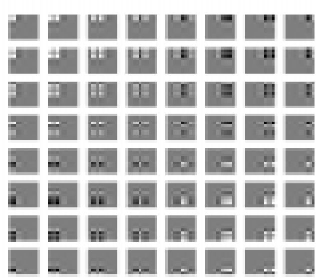

Figure 5

4. POTENTIAL APPLICATIONS

The time localization property of DFrCT has some potential applications for a number of specific waveforms and signal types. As an example, for chirp-like signals, the time localization property may exploit the changing spectral characteristics of the transform along the time domain.

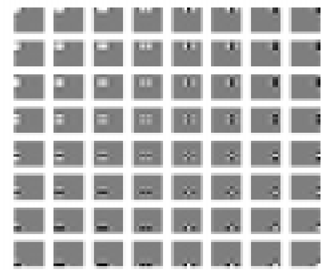

Figure 6

For normal images, where the signal can be represented as an AR(1) sequence with a high correlation, we know that DCT performs quite near to the KLT, which is the optimum average transform. For the same type of signals, the DFrCT performance gets degraded as the fraction index gets smaller (Fig. 7).

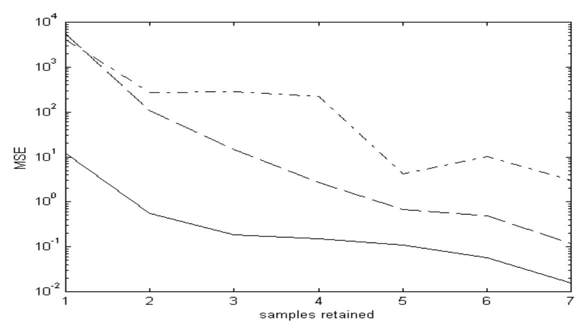

Figure 7

This figure shows the basis restriction error [12] plots in the logarithmic scale in terms of the number of samples retained in the transform domain for an AR(1) sequence with ρ=0.95. The basis restriction error is the mean squared error between the original signal and the transformed and inverse transformed signal after eliminating some of the transform coefficients and retaining the first N elements, where N, the number of retained elements, are indicated in the horizontal axis of Fig. 7. The solid line shows the regular DCT (or DFrCT at scale=1), the dashed line shows DFrCT at scale = 0.8, the dot-dashed line shows DFrCT at scale = 0.4.

However, for an arbitrary signal, the retained transform coefficients need not be in the order from the first to the last. Or, they need not be signals of an AR(1) nature. The wavelet theory and the appropriate coders is indeed an attempt to find the suitable transformations which yield significant transform coefficients in a localized manner. The coders and quantizers, then, exploit the localization, and obtain compact representations.

In the case of fractional Fourier domain, the signals which have localized distributions in the Wigner distribution, but don't have the localization neither in the time domain, nor in the frequency domain, can be represented in a very compact manner. The cosine transform in the fractional domain incorporates this property of the fractional Fourier transform together with the nice properties of the cosine transform, such as being real-valued. In order the DFrCT to have a value for coding, appropriate coders which exploit the energy localizations, should be found. As another interesting problem, the class of signals which have a tilted Wigner distribution can be investigated. The digital images which have strong perspective lines have chirp-like structures, and can be a potential application for the fractional domain processing.

5. CONCLUSIONS

In this paper we introduced a novel transform called DFrCT with an extra free parameter, its fraction, which reduces to DCT when its fraction is unity. With its extra free parameter this newly proposed transform may result in better performance figures in all applications where DCT is used.

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