POWER RATIO DEFINITIONS AND ANALYSIS IN SINGLE CARRIER MODULATIONS

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ABSTRACT

In the past few years, many research activities have concerned the Peak to Average Power Ratio (*PAPR*) especially in an Orthogonal Frequency Division Multiplexing (OFDM) context. Nevertheless, we have noticed that the *PAPR* definition is not always the same from one author to the other, depending on the paper context which leads to confusions and bad interpretations. That's why we propose in this article to generalize the *PAPR* definition by introducting the Power Ratio (*PR*) parameter from which we can derive all the possible *PAPR* versions. Thereby, we propose a theoretical analysis of the *PR* vs oversampling and the roll off factor in single carrier modulation when a Nyquist filter is considered. We give an upper bound of the *PR* and simulations for QPSK and 16QAM modulations.

1. INTRODUCTION

Power is a critical issue in telecommunications and many studies and papers have discussed on how important power efficiency is in a communication chain, especially for embedded systems. This point has been emphasized with new modulations schemes like OFDM for which the noise-like temporal waveform makes the use of non linear devices as power amplifiers very critical. To clearly identify those temporal signal fluctuations, the ratio of the instantaneous power to the mean power has been defined and derived in many versions as Peak to Average Power Ratio (*PAPR*), Crest Factor (CF), *PMEPR* (Peak to Mean Envelop Power Ratio), PEP (Peak Envelop Power), ... In fact, having a look in the literature, it is clear that there exist a great number of different definitions which lead to confusions in spite of the fact that they are very close to each other. In this paper, we will first propose in section 2 a generalized definition called the Power Ratio (*PR*). The link between the *PR* parameter and all the above definitions found in the literature will first be made. Then, section 3 will concern the *PR* analysis vs the oversampling and the roll off factor of a Nyquist filter with simulation results.

2. POWER RATIO DEFINITIONS

2.1 Theoretical definition

Several notations will be used to precisely define the conditions in which the *PR* is derived. The indices c (for continuous), s (for sampled), f (for finite) and i (for infinite) can be combined to express the four possible *PR* situations (see Fig. 1). The finite (resp. infinite) continuous *PR*, noted $PR_{c,f}$ (resp. $PR_{c,i}$) is used when the integration time is equal to T (resp. infinity).

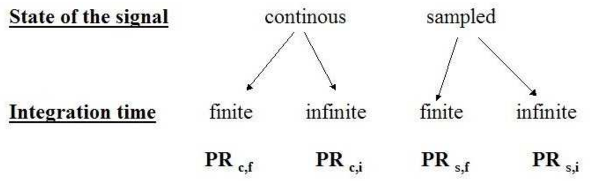

Figure 1: Different *PR* situations

2.1.1 *The infinite continuous $\mathbf{PR}_{c,i}$ definition*

To be coherent with the analog signal which needs to be amplified, the infinite *PR* is defined in a continuous case. $s(t)$ can be either a baseband, a Radio Frequency (RF), a real or a complex signal. In the case of a baseband signal (resp. RF), we will use the specific notation $\widetilde{PR}$ (resp. $\mathscr{PR}$). If the notation ***PR*** is used, the definition is valid in both cases. In those conditions,

$$\mathbf{PR}_{c,i}\{s(t)\} = \frac{Max|s(t)|^2}{E\{|s(t)|^2\}} = \frac{P_{max}}{P_{mean}}. \quad (1)$$

The variable t is real. $E\{.\}$ defines the expectation operation.

2.1.2 *The finite continuous $\mathbf{PR}_{c,f}$ definition*

$\mathbf{PR}_{c,f}$, for an integration time T, can be expressed as:

$$\mathbf{PR}_{c,f}(T)\{s(t)\} = \frac{Max_{[0,T]}|s(t)|^2}{\frac{1}{T}\int_T |s(t)|^2 dt}. \quad (2)$$

In those conditions, we have the following expression:

$$\lim_{T\to+} \mathbf{PR}_{c,f}(T) = \mathbf{PR}_{c,i}. \quad (3)$$

2.1.3 *The finite sampled $\mathbf{PR}_{s,f}$ definition*

In sampling conditions, we define N as the number of signal samples. Then, the finite sampled $\mathbf{PR}_{s,f}(N)$ is (the variable k is integer):

$$\mathbf{PR}_{s,f}(N)\{s(k)\} = \frac{Max_{k\in[0,N-1]}|s(k)|^2}{\frac{1}{N}\ {}_{k=0}^{N-1}|s(k)|^2}. \quad (4)$$

As before, we have the following relation:

$$\lim_{N\to} \mathbf{PR}_{s,f}(N) = \mathbf{PR}_{s,i}, \quad (5)$$

with

$$\mathbf{PR}_{s,i}\{s(k)\} = \frac{Max|s(k)|^2}{E\{|s(k)|^2\}}. \quad (6)$$

2.2 Usual Power Ratio definitions

In the following developments, the state of the signal is supposed to be continuous and the integration time be infinite. Neverthless, the other three combinations (see Fig. 1) are also valid.

2.2.1 *The PMEPR definition*

The *PMEPR* (Peak to Mean Envelop Power Ratio) is used when the basedband signal is considered (complex envelope) instead of the modulated signal. The *PMEPR* is then a derivation of the *PR* for baseband signals. This complex signal, $\widetilde{s}(t) = s_I(t) + js_Q(t)$, leads to the *PMEPR* definition [3]:

$$PMEPR\{\widetilde{s}(t)\} = \frac{Max|\widetilde{s}(t)|^2}{E\{|\widetilde{s}(t)|^2\}}. \quad (7)$$

In those conditions, $PMEPR = \widetilde{PR}_{c,i}$.

2.2.2 *The PAPR definition*

The *PAPR* (Peak to Average Power Ratio) is commonly defined as a derivation of the *PR* for RF signals. It can be written as:

$$PAPR\{\widetilde{s}(t)\} = \frac{Max|Re(\widetilde{s}(t)e^{j\ \ _0 t})|^2}{E\{|Re(\widetilde{s}(t)e^{j\ \ _0 t})|^2\}}. \quad (8)$$

In RF conditions, $PAPR = \mathscr{PR}_{c,i}$.

$_0$ is the carrier frequency. The *PAPR* is a measure of the signal fluctuations after the RF transposition (so just before the amplifier device). In [1], the authors put the stress on a definition of the *PAPR* using the envelope of the signal and in [2], the authors use *PMEPR* for dissociating it from *PAPR*. In [2], the authors propose a development of the relationship between *PMEPR* and *PAPR*.

2.2.3 *The CF definition*

The Crest Factor (CF) is sometimes used and is defined as the square root of either the *PAPR* or the *PMEPR* depending on the study case (RF or not). In [5], the square root of the *PAPR* is proposed and in [4], it is the square root of the *PMEPR* :

- $CF = \sqrt{PAPR} = \sqrt{\mathscr{PR}_{c,i}}$ in RF conditions
- $CF = \sqrt{PMEPR} = \sqrt{\widetilde{PR}_{c,i}}$ in baseband conditions

2.2.4 *The PEP definition*

The *PEP* (Peak Envelop Power) is more scarcely used and defines the maximum value of the instantaneous power. In [6], the *PEP* is defined by:

$$PEP\{s(t)\} = Max|s(t)^2|. \quad (9)$$

2.2.5 *The EPF definition*

As *PR* is a random variable, a way of investigation is to find its distribution function or an approximation, what could be a very hard task. To do so, some authors [1] propose to investigate the Effective Peak Factor (EPF) defined as:

$$Pr[\mathbf{PR} > EPF] = \quad (10)$$

where   is a negligible probability. The EPF leads to *PR* upper bounds distributions calculations.

3. POWER RATIO ANALYSIS IN SINGLE CARRIER MODULATION

To analyse the Power Ratio, we separate it into two subproblems. Firstly, in subsection 3.1 we study the oversampling influence, without Nyquist filter, i.e. with a window shape filter at the sampling of interest. Clearly this restriction is not realistic but is very useful for this oversampling influence analysis. Secondly, in subsection 3.2 in the continuous (analog) case, we analyze the Nyquist filter influence.

3.1 Oversampling influence

Starting from the QPSK modulation (Fig. 2) and computing the $PR_{s,i}$ at the symbol frequency we find a value of 0dB which does not correspond to the reality. Unfortunately, this could be found in some papers. But it is very easy to understand that such a value is not possible, because taking the transitions between the symbols into account, it is clear that during part of time, the power will not be constant. It will be half the max power with a probability of $\frac{1}{2}$ and it will be equal to 0 with a probability of $\frac{1}{4}$.

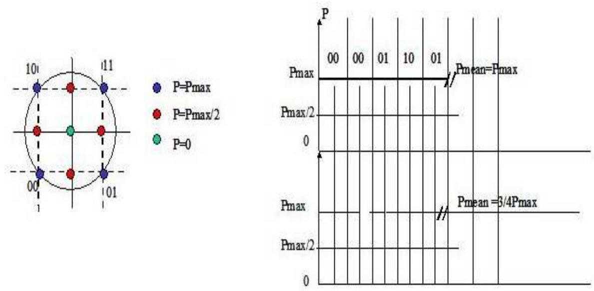

Figure 2: a) QPSK scattering diagram ; b) *PR* temporal variation in the T/2 case

As an example, if we assume an oversampling of 2 with an equal sharing of the symbol duration between the true symbol and the transition (see Fig. 2), we obtain a first approximation of the mean power P_{mean}, the maximal power being identical:

$$P_{mean} = \frac{P_{max}}{2} + \frac{1}{2} * (\frac{P_{max}}{4} + \frac{P_{max}}{2*2} + \frac{0}{4}). \quad (11)$$

Then the $PR = 1.25dB$. This value is still a lower bound of the true *PR*. In fact, we have to generalize the previous equation. We assume a sampling frequency of $\frac{N}{T}$. In those conditions, we have:

- a probability of $\frac{1}{4}$ for staying on the same symbol, so $P_0 = \frac{P_{max}}{4}$

- a probability of $\frac{1}{2}$ for going from one symbol to a neighbour one. Then the all power is given by:

$$P_N(i) = (\frac{1}{2} + \frac{(N/2-i)^2}{2*(N/2)^2})P_{max} \qquad i \in [0, N-1]. \quad (12)$$

The global contribution to the mean power is then:

$$P_1 = \frac{P_{max}}{2N} \sum_{i=0}^{N-1} \frac{(N/2-i)^2}{2*(N/2)^2} = \frac{1+2N^2}{6N^2} P_{max}. \quad (13)$$

- a probability of $\frac{1}{4}$ for going from one symbol to the diametric opposite. Then the all power is given by:

$$P_N(i) = (\frac{1}{2} + \frac{(N/2-i)^2}{2*(N/2)^2})P_{max} \qquad i \in [0, N-1]. \quad (14)$$

The global contribution to the mean power is then:

$$P_2 = \frac{P_{max}}{4N} \sum_{i=0}^{N-1} \frac{(N/2-i)^2}{(N/2)^2} = \frac{2+N^2}{12N^2} P_{max}. \quad (15)$$

As $P_{mean} = P_0 + P_1 + P_2$, we finally get $PR = \frac{3N^2}{1+2N^2}$ and the limit of the PR is 1.7609 dB which is an upper PR bound for the unfiltered QPSK. This bound is not the maximal and realistic upper bound because we did not take into account the Nyquist filter, what will be studied in the next section.

3.2 Nyquist filtering influence

In this section, the $PMEPR = \widetilde{PR}_{c,i}$ analysis of a single-carrier modulation is derived (in baseband) by considering a Nyquist filter *p(t)* depending on the roll off factor :

$$p(t) = \frac{sin(\frac{}{T_s}t)}{\frac{}{T_s}t} \frac{cos(\frac{}{T_s}t)}{1 - \frac{4\ ^2t^2}{T_s^2}}. \quad (16)$$

T_s is the symbol period and the $\widetilde{PR}_{c,i}$ will be analyzed with an infinite oversampling rate. To be coherent with the last section, the $\widetilde{PR}_{c,i}$ will be derived in an infinite way, that is for an infinite integration time ; the analyzed signal $\widetilde{s}(t)$ is expressed as:

$$\widetilde{s}(t) = \sum_{k=0}^{N-1} (a_k + jb_k)p(t - kT_s). \quad (17)$$

The following developments will respectively concern the theorical analysis of the mean and the maximum value of the instantaneous power $|\widetilde{s}(t)|^2$ in BPSK conditions, for $\widetilde{PR}_{c,i}\{\widetilde{s}(t)\}$ estimation with Nyquist filtering. Section 3.3 will concern other square modulations.

Mean power analysis in BPSK: the mean output power P_m may be expressed as:

$$P_m = E\{|\widetilde{s}(t)|^2\} = \int_{-} \ _e(f)P^2(f)df, \quad (18)$$

where $_e(f)$ is the spectral power density function of $e(t) = \sum_{k=0}^{N-1} a_k \ (t - kT_s)$ and *P(f)* the Fourier transform of *p(t)*. Assuming uncorrelated symbols with zero mean, $_e(f) = \frac{E\{a_k^2\}}{T_s} = \frac{^2_a}{T_s}$.

$P(f)$ is equal to T_s, $\frac{T_s}{2} + \frac{T_s}{2} sin(\frac{T_s}{}(\frac{1}{2T_s} - |f|)$, and 0 respectively for $|f| \leq \frac{1-}{2T_s}$, $\frac{1-}{2T_s} \leq |f| \leq \frac{1+}{2T_s}$ and $|f| \geq \frac{1+}{2T_s}$. Then, the integration of $P^2(f)$ gives:

$$\int_{-} P^2(f)df = T_s(1 - \frac{}{4}). \quad (19)$$

Then, $P_m = \ ^2_a(1 - \frac{}{4})$.

Maximum power analysis in BPSK: for this analysis, we first have to study the case $= 0$ otherwise, (27) will not be valid.
Then, for $= 0$, if we take $a_k = (-1)^k$ and $t = \frac{-T_s}{2}$, we get

$$\widetilde{s}(t = -\frac{T_s}{2}) = \frac{2}{} \sum_{k=0}^{N-1} \frac{1}{1+2k}, \quad (20)$$

and it is clear that this series diverges when N tends to infinity. Then,

$$\widetilde{PR}_{c,i}\{\widetilde{s}(t), \ = 0\} = + \ . \quad (21)$$

Let us now analyze the behavior of $|\widetilde{s}(t)|^2$ for $\neq 0$. For a better legibility, we normalized the symbol rate:

$$\widetilde{s}(t) = \sum_{k=0}^{N-1} a_k \frac{sin(\ (t-k))}{(t-k)} \frac{cos(\ (t-k))}{1-4\ ^2(t-k)^2}. \quad (22)$$

From the following inequality,

$$|\frac{sin(\ (t-k))}{(t-k)} \frac{cos(\ (t-k))}{1-4\ ^2(t-k)^2}| \leq \frac{1}{|1-4\ ^2(t-k)^2|}. \quad (23)$$

The right term of the inequality for any t is equivalent to a Rieman series when k tends to infinity. So, for $\neq 0$, the series converges absolutely what implies that the series converges. Then,

$$\widetilde{PR}_{c,i}\{\widetilde{s}(t), \ \neq 0\} \neq + \ . \quad (24)$$

Let us now investigate an upper bound of $|\widetilde{s}(t)|^2$. $|\widetilde{s}(t)|^2$ may be upper bounded by

$$\sum_{k=0}^{N-1} (\frac{sin(\ (t-k))}{(t-k)})^2 \sum_{k=0}^{N-1} (\frac{cos(\ (t-k))}{1-4\ ^2(t-k)^2})^2. \quad (25)$$

First, after some maths, it can be shown that the left term $\sum_{k=0}^{N-1} (\frac{sin(\ (t-k))}{(t-k)})^2$ of (25) can be upper bounded by 1.
Let us now analyze the right term: $(t) = \sum_{k=0}^{N-1} (\frac{cos(\ (t-k))}{1-4\ ^2(t-k)^2})^2$. It can be shown that is maximum for $t = \frac{N}{2}$. This is illustrated in Fig. 3 where the upper curve is the result of the sum of the N shifted functions $(\frac{cos(\ (t-k))}{1-4\ ^2(t-k)^2})^2$.

At $t = \frac{N}{2}$, it can be shown that

$$(t = \frac{N}{2}) = 1 + 2 \sum_{k=1}^{N/2} (\frac{cos(\ k)}{1-4\ ^2k^2})^2. \quad (26)$$

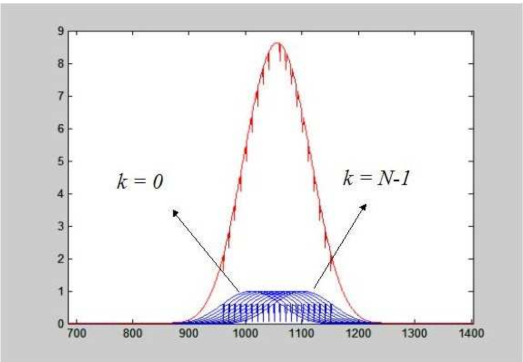

Figure 3: fonction (t)

The limit of this series can be investigated by using the general Euler-Mac Laurin relation ($_{k=1}^{n} F(k) = A + B$) where $A = \int_0^n F(t)dt$, $B = -\frac{1}{2}(F(0) - F(n)) + \frac{1}{2}(F'(n) - F'(0)) + \ldots$ and $F(k) = (\frac{cos(\quad k)}{1-4 \quad ^2k^2})^2$. It can be shown (with the help of the residue theorem) that

$$\lim_{n\to} \int_0^n (\frac{cos(\quad t)}{1-4 \quad ^2t^2})^2 dt = \frac{2}{16} \qquad \in]0,1]. \qquad (27)$$

Moreover, $F(0) = 1$, $F(n) = 0$ and $F^{(p)}(n) = F^{(p)}(0) = 0$ when n tends to infinity and whatever the successive derivations order p of the of $F(k)$. Then,

$$|\widetilde{s}(t)|^2 \leq \frac{2}{8} \quad \Rightarrow \quad \widetilde{PR}_{c,i}\{\widetilde{s}(t)\}_{BPSK} \leq \frac{^2/8}{(1- \quad /4)}. \qquad (28)$$

3.3 Analysis for other digital modulations

From (17), as a_k and b_k have the same statistics, we can easily get the following general $\widetilde{PR}_{c,i}$ expressions for QAM digital modulations:

$$\widetilde{PR}_{c,i}\{\widetilde{s}(t)\}_{QAM} \leq (\frac{Max(a_k)}{a})^2 \frac{^2/8}{(1- \quad /4)}. \qquad (29)$$

3.4 Upper bounds comparisons with simulations

Fig. 4 illustrates the simulation for both QPSK and 16QAM modulations. For the larger values of (0.9 and 0.6), the computation has been led on the base of 10^4 loops of $N = 1000$ symbols each to get the maximum $\widetilde{PR}_{c,i}$ value. The similitude between theory and simulations is excellent. For smaller roll off values (0.3 and 0.1), we have 10^6 loops of $N = 10^5$ symbols each. This can be explained in part because of the statistical comportment of the infinite $\widetilde{PR}_{c,i}$: when tends to zero, we have observe that the $\widetilde{PR}_{c,i}$ probability density function is very dispersive and getting the infinite $\widetilde{PR}_{c,i}$ value needs large computations whereas, for larger roll off values, the probability law is more concentrated.

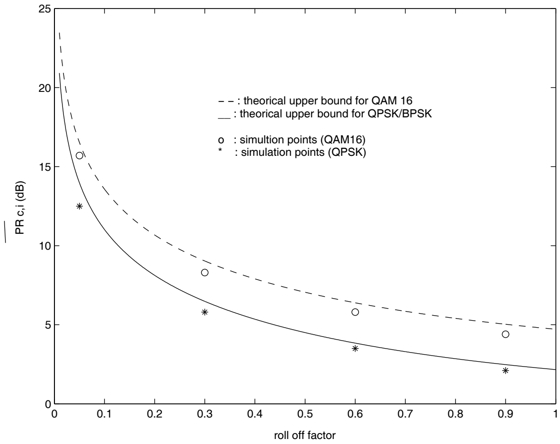

Figure 4: The $\widetilde{PR}_{c,i}$ vs roll off factor

4. CONCLUSION

In this paper, we first propose a new definition called the Power Ratio (*PR*) which generalized all the well known classical cases as *PAPR*, *PMEPR*, *PEP*, ... The *PR* is derived in single carrier vs the oversampling and vs the roll off factor of a Nyquist filter for which we propose new bounds and show that simulation results are very close to theory in both QPSK and 16QAM modulations. Further investigations will concern multicarrier modulations and their *PR* analysis vs oversampling and roll off factor.

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