* Clearly justify all answers.
* All angles are in radians.

1) An analog signal \( x_a(t) \), is given as:
\[
x_a(t) = 2 \sin t + 4 \cos(t-2) + 3 \cos(3t-1) + 5
\]
\( \text{in seconds} \)

5 pts a) Plot its spectrum

4 pts b) \( x_a(t) \) is sampled, with a sampling interval of \( T_s = 3 \) sec; find the discrete signal \( x[n] \). Is there aliasing?

6 pts c) Knowing that the analog signal consists of sinusoidal components, and assuming that there were no aliasing, someone recovers an \( x_r(t) \) from \( x[n] \), with \( T_s = 3 \) sec.
Find \( x_r(t) \). Are \( x_a(t) \) and \( x_r(t) \) the same?

4 pts d) Is \( x_a(t) \) periodic? If "yes" find its period.

6 pts e) Find the Fourier series expansion of \( x_a(t) \).
Find the Fourier series expansion of $x_a(t)$, where,

$$x_a(t) = \begin{cases} 1 & -\frac{1}{12} \leq t \leq \frac{1}{12} \\ 0 & -\frac{1}{12} \leq t \leq -\frac{1}{12}, \frac{1}{12} < t \leq 1 \end{cases}$$

and $x_a(t)$ is periodic with $T = 2$ sec.

2 pts b) Plot $x_a(t)$.

10 pts c) Find the Fourier transform of $y_a(t)$, where

$$y_a(t) = \begin{array}{c} \text{Periodic} \\ \text{T=2 sec} \\ \text{One period} \\ T=2 \sec \end{array}$$

(Hint: Use your answer to 2-a)
3-) A linear and shift invariant system is known to generate the output $y_a(t)$ to $x_a(t)$ where

\[ x_a(t) : \]

\[ y_a(t) : \]

Find its output if the input is

\[ x_1(t) : \]

4-) Check if the following systems are causal, linear, shift invariant:

6 pts  i) $y[n] = (n+1) \cdot (x[n] - x[n-1])$

6 pts  ii) $y[n] = x[2n+1]$
5-)
An LTI FIR system is given as:

\[ y[n] = \frac{1}{2} x[n+1] - x[n] + \frac{1}{2} x[n-1] \]

2 pts a) Find its impulse response

10 pts b) Find its frequency response
Plot the magnitude and phase of the frequency response.

3 pts c) What kind of a filter is this?

8 pts d) Find \( y[n] \) if

\[ x[n] = 3 + (-1)^n + \frac{1}{2} \cos \frac{\pi}{4} n \]
1) \( x_a(t) = 2 \sin t + 4 \cos (t-2) + 3 \cos (3t-1) + 5 \)

a) Using the identities
\[
\cos(t) = \frac{e^{jt} + e^{-jt}}{2} \quad \sin(t) = \frac{e^{jt} - e^{-jt}}{2j}
\]
we get
\[
x_a(t) = \frac{4}{j} e^{jt} - \frac{4}{j} e^{-jt} + 2 e^{j(t-2)} + 2 e^{-j(t-2)} + \frac{3}{2} e^{j(3t-1)} + \frac{3}{2} e^{-j(3t-1)} + 5
\]

Grouping terms of same angular frequency together, we get
\[
x_a(t) = \left( \frac{4}{j} + 2 e^{-2j} \right) e^{jt} + \left( 2 e^{2j} - \frac{4}{j} \right) e^{-jt} + \frac{3}{2} e^{-j} e^{j3t} + \frac{3}{2} e^{j} e^{-j3t} + 5 e^{j0}
\]
b) $x[n] = x_a(nT_s)$

$T_s = 3$ sec

$\Rightarrow x[n] = x_a(nT_s) = x_a(3n)$

$\Rightarrow x[n] = 2\sin(3n) + 4\cos(3n-2) + 3\cos(9n-1) + 5$

$\omega_{\text{max}} \rightarrow$ the maximum angular frequency contained by $x_a(t)$.

$\omega_s = \frac{2\pi}{T_s} \rightarrow$ sampling angular frequency

* If $\omega_{\text{max}} > \frac{\omega_s}{2}$, there is ALIASING.

In our case:

$\omega_{\text{max}} = 3 \frac{\text{rad}}{\text{sec}}$ (Due to the $3\cos(3t-1)$ term)

$\omega_s = \frac{2\pi}{3} \frac{\text{rad}}{\text{sec}}$

Since $\omega_{\text{max}} = 3 \frac{\text{rad}}{\text{sec}} > \frac{\omega_s}{2} = \frac{\pi}{3} \frac{\text{rad}}{\text{sec}}$

there is aliasing.
c) Let us first reduce the angular frequencies of sinusoids within \( x[n] \) to \([-\pi, \pi]\) interval. The important is to notice that
\[
\cos((\omega_0 + 2\pi k)n + \varphi) = \cos(\omega_0 n + \varphi + 2\pi k n) = \cos(\omega_0 n + \varphi) \quad \text{for any integer } k.
\]
Refer to the expression for \( x[n] \) at part b.
\[
\sin(3n) \quad \rightarrow \quad \omega_0 = 3 \text{ rad}, \; \omega_0 \in [-\pi, \pi) \quad \checkmark
\]
\[
\cos(3n-2) \quad \rightarrow \quad \omega_0 = 3 \text{ rad}, \; \omega_0 \in [-\pi, \pi) \quad \checkmark
\]
\[
\cos(9n-1) \quad \rightarrow \quad \omega_0 = 9 \text{ rad}, \; \omega_0 \in [-\pi, \pi) \quad \text{but we know that}
\]
\[
\cos((9 + 2\pi k)n - 1) = \cos((9n - 1), \quad \text{select } k = 1
\]
\[
\cos(8n - 1) = \cos((9 - 2\pi)n - 1) \quad \text{with } (9 - 2\pi \text{ rad}) \in [-\pi, \pi) \quad \checkmark
\]
\[
\Rightarrow \quad x[n] = 2 \sin(3n) + 4 \cos(3n-2) + 3\cos((9-2\pi)n-1) + 5
\]
\[
\Rightarrow \quad \text{In this expression, all angular frequencies are in the } [-\pi, \pi) \text{ interval.}
\]

* When we sample \( x_R(t) \) and if there is no ALIASING, we should have \( \omega_{\text{max}}^R \leq \frac{\omega_s}{2} \). (\( \omega_{\text{max}}^R \rightarrow \text{maximum angular frequency contained in } x_R(t) \).

We are also told that \( x_R(t) \) consists of sinusoidal components. Suppose \( \omega_{\text{max}}^R \leq \frac{\omega_s}{2} \) is satisfied, and suppose \( \alpha \cos(\omega_0 t + \varphi) \) is one of the sinusoidal terms in \( x_R(t) \). (\( \omega_0 \leq \frac{\omega_s}{2} \), by the assumption \( \omega_{\text{max}}^R \leq \frac{\omega_s}{2} \)), when we sample \( \alpha \cos(\omega_0 t + \varphi) \) with \( T_s = \frac{2\pi}{\omega_s} \), we get the discrete signal
\[
\alpha \cos(\omega_0 nT_s + \varphi) = \alpha \cos(\frac{2\pi}{\omega_s} n + \varphi)
\]
\[
\text{Note that the angular frequency of this sinusoid } \frac{2\pi}{\omega_s} \text{ is already in the } [-\pi, \pi) \text{ interval since } \omega_0 \leq \frac{\omega_s}{2}.\]
Now knowing that there is no aliasing, we should ask the following question:

What should be \( w_o \) so that when we sample \( 2 \sin (w_o t) \) with \( T_s = \frac{2\pi}{w_s} = 3 \text{ sec} \); we get the discrete signal \( 2 \sin (3 \pi n) \); such that \( w_o \leq \frac{w_s}{2} \)?

Let \( \frac{2\pi w_o}{w_s} = 3 \) \( \Rightarrow \) \( w_o = \frac{3w_s}{2\pi} = \frac{3}{2\pi} \left( \frac{2\pi}{3} \text{ rad/sec} \right) = 1 \text{ rad/sec} \)

\( \Rightarrow \) If we sample \( 2 \sin (t) \) with \( T_s = 3 \text{ sec} \); we get the discrete signal \( 2 \sin (3 \pi n) \); such that \( w_o = 1 \text{ rad/sec} \leq \frac{w_s}{2} = \frac{\pi}{3} \text{ rad/sec} \) \( \Rightarrow \) there is NO ALIASING.

Similarly, we see that

\[ 4 \cos \left( t - \frac{2\pi}{3} \right) \text{ sample with } T_s = 3 \] \[ 4 \cos \left( 3 \pi n - \frac{2\pi}{3} \right) \]

\[ 3 \cos \left( \left( \frac{9-2\pi}{3} \right) t - 1 \right) \text{ sample with } T_s = 3 \]

\[ 3 \cos \left( \left( \frac{9-2\pi}{3} \right) n - 1 \right) = 3 \cos (3 \pi n - 1) \]

with no aliasing.

\[ \Rightarrow X_R (t) = 2 \sin t + 4 \cos \left( t - \frac{2\pi}{3} \right) + 3 \cos \left( \left( \frac{9-2\pi}{3} \right) t - 1 \right) + 5 \]

\[ X_R (t) \neq X_a (t) \text{ due to the underlined term; though } X_R (3n) = X_a (3n) \text{. There is aliasing in } X_a (3n) \text{; but there is no aliasing in } X_R (3n). \]
d) \( x_R(t) \) is periodic if there exists \( T > 0 \) such that
\[
x_R(t + T) = x_R(t).
\]
In our case; if we select \( T = 2\pi \) sec; we see that
\[
x_a(t + 2\pi) = 2 \sin (t + 2\pi) + 4 \cos (t + 2\pi - 2) + 3 \cos (3t + 6\pi - 1) + 5
\]
Since \( \cos (\theta + 2\pi k) = \cos (\theta) \) for any integer \( k \),
we get
\[
\begin{align*}
\sin (t + 2\pi) &= \sin (t) \\
\cos (t + 2\pi - 2) &= \cos (t - 2) \\
\cos (3t + 6\pi - 1) &= \cos (3t - 1)
\end{align*}
\]
\( \Rightarrow x_a(t + 2\pi) = x_a(t) \Rightarrow x_a(t) \) is periodic.
Moreover; there exists no \( T \) smaller than \( 2\pi \) sec; so the fundamental period of \( x_a(t) \) is
\( T = 2\pi \) sec.

e) In Fourier series expansion, a signal \( x(t) \) with period \( T \) is written as a sum of complex sinusoids periodic with \( T \) as:
\[
x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\frac{2\pi j k t}{T}} \quad \text{with} \quad \alpha_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(z) e^{-\frac{2\pi j k z}{T}} \, dz.
\]
Here; we are actually asked to find \( \alpha_k \)'s. We can use the second formula; but here there is an easier way. Refer to part a where we plotted the spectrum of \( x_a(t) \). Examine the expression just above the plot. Insert \( T = 2\pi \) sec as found in part d; \( \Rightarrow x_a(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\frac{2\pi j k t}{T}} \) \( \Rightarrow \) \( \alpha_1 = \left( \frac{1}{2} + 2e^{-2j} \right) \). In this manner; you can see that\(\alpha_{-3} = \frac{3}{2} e^j, \quad \alpha_{-2} = 2e^{2j} - \frac{1}{3}, \quad \alpha_0 = 5, \quad \alpha_1 = \frac{1}{2} + 2e^{-j}, \quad \alpha_3 = \frac{3}{2} e^{-j}\)

and the rest of \( \alpha_k \)'s are 0.
2a) \[ x_a(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\frac{2\pi i k t}{T}} \] with \[ \alpha_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_a(z) e^{-\frac{2\pi i k z}{T}} dz \]

In our case, with \( T = 2 \) sec:

\[ \alpha_k = \frac{1}{2} \int_{-\frac{1}{12}}^{\frac{1}{12}} e^{-\frac{2\pi i k z}{T}} dz \]

\[ \Rightarrow \text{If } k = 0 \quad \alpha_k = \frac{1}{2} \int_{-\frac{1}{12}}^{\frac{1}{12}} dz = \frac{1}{12} \] (Do not forget to handle \( k=0 \) case separately)

\[ \Rightarrow \text{If } k \neq 0 \quad \alpha_k = \frac{1}{2} e^{-\frac{2\pi i k}{T}} \int_{-\frac{1}{12}}^{\frac{1}{12}} dz = \frac{1}{12} \left( e^{\frac{2\pi i k}{12}} - e^{-\frac{2\pi i k}{12}} \right) = \sin \left( \frac{\pi k}{12} \right) \]

Noting that \( \text{sinc}(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0 \end{cases} \)

we can write \[ \alpha_k = \frac{1}{12} \text{sinc} \left( \frac{k}{12} \right) \]

b)
c) Here, the most important point is to recognize that

\[ y_a(t) = \frac{1}{2} x_a(t) + \frac{1}{4} x_a(t - \frac{3}{12}) \]

(Think on this equation; and try to understand it with no confusion.)

After recognizing this; (which also implies that \( y_a(t) \) has the same period \( T = 2 \text{ sec} \) with \( x_a(t) \)); note that we are trying to find \( \beta_k \)'s such that

\[ y_a(t) = \sum_{k=-\infty}^{\infty} \beta_k e^{\frac{2 \pi i t k}{T}} \]

In part a, we already found \( \alpha_k \)'s for

\[ x_a(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\frac{2 \pi i t k}{T}}. \]

Now, let us try to relate \( \beta_k \)'s to \( \alpha_k \)'s using

\[ y_a(t) = \frac{1}{2} x_a(t) + \frac{1}{4} x_a(t - \frac{3}{12}) \]

\[ \Rightarrow y_a(t) = \sum_{k=-\infty}^{\infty} \frac{\alpha_k}{2} e^{-\frac{2 \pi i t k}{T}} + \sum_{k=-\infty}^{\infty} \frac{\alpha_k}{4} e^{\frac{2 \pi i t (k - \frac{3}{12})}{T}} \]

\[ \Rightarrow y_a(t) = \sum_{k=-\infty}^{\infty} \frac{\alpha_k}{2} e^{-\frac{2 \pi i t k}{12T}} + \sum_{k=-\infty}^{\infty} \frac{\alpha_k}{4} e^{-\frac{2 \pi i t (k - \frac{3}{12})}{12T}} e^{\frac{2 \pi i t k}{12T}} \]

\[ \Rightarrow y_a(t) = \sum_{k=-\infty}^{\infty} \frac{\alpha_k}{2} \left(1 - e^{-\frac{2 \pi i k}{12T}}\right) e^{\frac{2 \pi i t k}{12T}} \]

with \( T = 2 \text{ sec} \)

\[ \beta_k = \frac{\alpha_k}{2} \left(1 - e^{-\frac{2 \pi i k}{4}}\right) \]

with \( \alpha_k = \frac{1}{12} \text{sinc}\left(\frac{k}{12}\right) \)
3) Here, you should first recognize that

\[ x(t) = x_a(t) + x_a(t-1) \]

Next, the shift invariance property of the system implies that

\[ x_a(t-1) \xrightarrow{\text{generates}} y_a(t-1) \]

and the linearity of the system implies that

\[ x_a(t) + x_a(t-1) \xrightarrow{\text{generates}} y_a(t) + y_a(t-1) \]

\[ \Rightarrow y(t) = y_a(t) + y_a(t-1) \]
4) Let $S$ denote our system: $y[n] = S \{ x[n] \}$

**Linearity:**

$y_1[n] = S \{ x_1[n] \}$

$y_2[n] = S \{ x_2[n] \}$

If $S \{ ax_1[n] + bx_2[n] \} = ay_1[n] + by_2[n]$ for all $a, b, x_1[n], x_2[n]$; the system is linear.

**Shift-invariance:**

$y[n] = S \{ x[n] \}$

If $S \{ x[n-n_0] \} = y[n-n_0]$ for all $x[n], n_0$; the system is shift invariant.

**Causality:**

Let $x_1[n]$ and $x_2[n]$ be two inputs such that

$x_1[n] = x_2[n]$ for $n < n_0$.

(But $x_1[n] \neq x_2[n]$ for $n > n_0$ is possible)

Let $y_1[n] = S \{ x_1[n] \}$, $y_2[n] = S \{ x_2[n] \}$

If $y_1[n] = y_2[n]$ for $n < n_0$ (and if this holds for all $x_1[n], x_2[n]$ satisfying $x_1[n] = x_2[n]$ for $n < n_0$ and for all $n_0$); $S$ is causal.

Another equivalent statement for causality is: $S$ is causal if at any time instant $n$; $y[n]$ depends on past values $x[n], x[n-1], x[n-2], \ldots$; but not on future values $x[n+1], x[n+2], \ldots$, etc.
i) **Linearity:**

Let \( S\{x_1[n]\} = y_1[n] \) , \( S\{x_2[n]\} = y_2[n] \)

\[ y_1[n] = (n+1)(x_1[n] - x_1[n-1]) \]
\[ y_2[n] = (n+1)(x_2[n] - x_2[n-1]) \]

Let \( x_3[n] = a x_1[n] + b x_2[n] \)

\[ y_3[n] = S\{x_3[n]\} \]

\[ y_3[n] = (n+1)(x_3[n] - x_3[n-1]) \]
\[ = (n+1)(ax_1[n] + bx_2[n] - ax_1[n-1] - bx_2[n-1]) \]
\[ = (n+1)(ax_1[n] - ax_1[n-1]) + (n+1)(bx_2[n] - bx_2[n-1]) \]
\[ = a(n+1)(x_1[n] - x_1[n-1]) + b(n+1)(x_2[n] - x_2[n-1]) \]

\[ y_3[n] = ay_1[n] + by_2[n] \Rightarrow S \text{ is linear.} \]

**Shift-invariance:**

Let \( S\{x[n]\} = y[n] \) , \( x_4[n] = x[n-n_0] \) ; and \( y_1[n] = S\{x_4[n]\} \)

\( S \) is shift-invariant if \( y_1[n] = y[n-n_0] \).

\[ y_1[n] = S\{x_4[n]\} = (n+1)(x_1[n] - x_1[n-1]) \]
\[ = (n+1)(x[n-n_0] - x[n-n_0-1]) \]

On the other hand;

\[ y[n] = (n+1)(x[n] - x[n-1]) \]
\[ y[n-n_0] = (n-n_0+1)(x[n-n_0] - x[n-n_0-1]) \]

Since \( y_1[n] \neq y[n-n_0] \); \( S \) is not shift invariant.

**Causality:**

Computation of \( y[n] \) requires the knowledge of \( x[n] \) at \( n_0 \) and \( n_0-1 \) (past values); so the system is causal.
ii) **Linearity:**

Let \( S \{ x_1[n] \} = y_1[n] \), \( S \{ x_2[n] \} = y_2[n] \)

\[
\Rightarrow y_1[n] = x_1[2n+1], \quad y_2[n] = x_2[2n+1]
\]

If \( x_3[n] = ax_1[n] + bx_2[n] \),

\[
S \{ x_3[n] \} = x_3[2n+1] = ax_1[2n+1] + bx_2[2n+1] = ay_1[n] + by_2[n]
\]

\[
\Rightarrow S \{ ax_1[n] + bx_2[n] \} = aS \{ x_1[n] \} + bS \{ x_2[n] \}
\]

\( \Rightarrow \) system is linear.

**Shift-invariance:**

Let \( S \{ x[n] \} = y[n] \), \( x[n] = x[n-n_0] \), \( y_1[n] = S \{ x_1[n] \} \)

\[
\Rightarrow y[n] = x[2n+1] \Rightarrow y[n-n_0] = x[2(n-n_0)+1] = x[2n+1]
\]

On the other hand,
\[
y_1[n] = x_1[2n+1] = x[2n+1-n_0]
\]

Since \( y_1[n] \neq y[n] \); system is not shift-invariant.

**Causality:**

Computation of \( y[n_0] \) requires the knowledge of \( x[n] \) at \( n = 2n_0+1 \). For instance, to compute \( y[1] \), we need \( x[3] \) (a future value). Hence;

\( S \) is not causal.
5) a) "Impulse response" of a system means the output of the system to an impulse input \( x[n] = \delta[n] \).

If we denote impulse response by \( h[n] \);
\[ h[n] = \mathcal{F}\{ \delta[n] \} \]

Using the provided input/output (i/o) relation, we get
\[ h[n] = \frac{1}{2} \delta[n+1] - \delta[n] + \frac{1}{2} \delta[n-1] \]

b) Since our system is LTI, we know that if \( x[n] = e^{jn\omega} \); the output has the form \( y[n] = H(\omega) e^{jn\omega} \). In other words,

\[ x[n] = e^{jn\omega} \rightarrow \text{LTI system} \rightarrow y[n] = H(\omega) e^{jn\omega} \]

* If \( x[n] = e^{jn\omega} \):
\[ y[n] = \frac{1}{2} x[n+1] - x[n] + \frac{1}{2} x[n-1] \]
\[ \Rightarrow y[n] = \frac{1}{2} e^{jn\omega(n+1)} - e^{jn\omega} + \frac{1}{2} e^{jn\omega(n-1)} \]
\[ \Rightarrow y[n] = \left( \frac{1}{2} e^{jn\omega} - 1 + \frac{1}{2} e^{-jn\omega} \right) e^{jn\omega} \]
\[ \Rightarrow H(e^{jn\omega}) = \frac{1}{2} (e^{jn\omega} + e^{-jn\omega}) - 1 \]
\[ H(e^{jn\omega}) = \cos(\omega) - 1 \]
Note that $H(e^{jw})$ is real valued; but 
\[
\cos(w) - 1 \leq 0 \text{ for all } w.
\]

$$|H(e^{jw})| = 1 - \cos(w)$$

$$\frac{1}{H(e^{jw})} = \frac{1}{1 - \cos(w)}$$

(Note: $H(e^{jw})$ is always a periodic function of $w$ with period $2\pi$. Why? Do not forget to indicate this on your plots.)

c) Examining the magnitude response within the $[-\pi, \pi]$ interval; we see that this filter attenuates low frequency components (around $w = 0$) while passing frequency components around $w = \pm \pi$ (high frequency components) so it is a high pass filter.

d) The easiest way to find $y[n]$ is to recognize that if $x[n] = e^{j\omega_0 n}$, $y[n] = H(e^{j\omega_0}) e^{j\omega_0 n}$. In our case:

$$x[n] = 3e^{3\pi n} + e^{j\pi n} + \frac{1}{4} e^{j\frac{\pi}{4} n} + \frac{1}{4} e^{-j\frac{3\pi}{4} n}$$

$$\Rightarrow y[n] = H(e^{j\pi}) 3e^{3\pi n} + H(e^{j\pi}) e^{j\pi n} + \frac{1}{4} H(e^{j\frac{\pi}{4}}) e^{j\frac{\pi}{4} n} + \frac{1}{4} H(e^{-j\frac{3\pi}{4}}) e^{-j\frac{3\pi}{4} n}$$

Noting that $H(e^{j\omega}) = \cos(w) - 1$; we get

$$y[n] = -2 (-1)^n + \frac{1}{2} \left( \frac{12}{12} - 1 \right) \cos \left( \frac{\pi}{4} n \right)$$