Instructions: No notes or books are allowed. For full credit, provide complete details of your work.

Q1. Given \( x_1(t) \) and \( x_2(t) \) which are both periodic with \( T \), find the Fourier series coefficients of the following signals in terms of the Fourier series coefficients of \( x_1(t) \) and \( x_2(t) \).

(a) \( x(t) = x_1(t) + x_2(t) \)

(b) \( x(t) = \alpha x_1(t) + \beta x_2(t) - 2 \)

(c) \( x(t) = x_1^T(t) \cdot x_2(t) \)

(d) \( x(t) = \int_0^t x_1(z) x_2(t-z) \, dz \).

We can write

\[
x_1(t) = \sum_{k=-\infty}^{\infty} X_1[k] e^{j \frac{2\pi k t}{T}}, \quad x_2(t) = \sum_{k=-\infty}^{\infty} X_2[k] e^{j \frac{2\pi k t}{T}}
\]

where \( X_1[k] \) and \( X_2[k] \) denote the Fourier series coefficients of \( x_1(t) \) and \( x_2(t) \), respectively. We wish to find \( X[k] \) such that \( x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j \frac{2\pi k t}{T}} \).

(a) \( x(t) = x_1(t) + x_2(t) \)

\[
x(t) = \sum_{k=-\infty}^{\infty} X_1[k] e^{j \frac{2\pi k t}{T}} + \sum_{k=-\infty}^{\infty} X_2[k] e^{j \frac{2\pi k t}{T}}
\]

\[
x(t) = \sum_{k=-\infty}^{\infty} (X_1[k] + X_2[k]) e^{j \frac{2\pi k t}{T}} = \sum_{k=-\infty}^{\infty} X[k] e^{j \frac{2\pi k t}{T}}
\]

\[\Rightarrow X[k] = X_1[k] + X_2[k]\]
b) \[ x(t) = \alpha x_1(t) + \beta x_2(t) - 2 \]
\[ = \alpha \sum_{k=-\infty}^{\infty} X_1[k] e^{\frac{j2\pi k t}{T}} + \beta \sum_{l=-\infty}^{\infty} X_2[l] e^{\frac{j2\pi l t}{T}} \]
\[ = \alpha \sum_{k=-\infty}^{\infty} X_1[k] e^{\frac{j2\pi k t}{T}} + \beta \sum_{l=-\infty}^{\infty} X_2[l] e^{\frac{j2\pi l t}{T}} \]
\[ = \sum_{k=-\infty}^{\infty} (\alpha X_1[k] + \beta X_2[k]) e^{\frac{j2\pi k t}{T}} \]
\[ \Rightarrow X[k] = \alpha X_1[k] + \beta X_2[k] e^{-\frac{j2\pi k t}{T}} \]

\[ c) x(t) = x_1(t) x_2(t) \]
\[ = \left( \sum_{k=-\infty}^{\infty} X_1[k] e^{\frac{j2\pi k t}{T}} \right) \left( \sum_{l=-\infty}^{\infty} X_2[l] e^{\frac{j2\pi l t}{T}} \right) \]
\[ = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} X_1[k] X_2[l] e^{\frac{j2\pi (k+l)t}{T}} \]
\[ = \sum_{k=-\infty}^{\infty} X_1[k] \sum_{l=-\infty}^{\infty} X_2[l] e^{\frac{j2\pi (k+l)t}{T}} \]
\[ \text{let } k + l = m \]
\[ \Rightarrow l = m - k \]
\[ = \sum_{k=-\infty}^{\infty} X_1[k] \sum_{m=-\infty}^{\infty} X_2[m-k] e^{\frac{j2\pi m t}{T}} \]
\[ = \sum_{m=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} X_1[k] X_2[m-k] \right) e^{\frac{j2\pi m t}{T}} \]
\[ \Rightarrow X[m] = \sum_{l=-\infty}^{\infty} X_1[l] X_2[k-l] \]
\[ x(t) = \sum_{k=-\infty}^{\infty} X_1[k] \cdot X_2[k] \cdot e^{j \frac{2\pi kt}{T}} \]

\[ X[k] = T \cdot X_1[k] \cdot X_2[k] \]
Q2 Given a periodic signal $x(t)$ with period $T$, we obtain uniformly spaced samples of $x(t)$ with a sampling interval $T_s$: $x[n] = x(nT_s)$. Then we reconstruct a continuous time signal $\hat{x}(t)$ from these samples by using Shannon or sinc interpolation based D-to-C converter. (8pts) (a) Find the condition on $x(t)$ such that $\hat{x}(t) = x(t)$.

(12pts) (b) Is it true that $\hat{x}(t)$ is always periodic? If yes provide a proof. If no, provide a counter example.

\[
x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kn}{T}}
\]

If we want to get $\hat{x}(t) = x(t)$ using Shannon (ideal bandlimited) interpolation; we should satisfy the Nyquist criteria. This means we should have

$$|f_{\text{max}}| \leq \frac{f_s}{2}$$

where $|f_{\text{max}}|$ denotes the highest frequency contained in $x(t)$. Hence; we should have

\[
X[k] = 0 \quad \text{whenever} \quad \left| \frac{k}{T} \right| > \frac{1}{2T_s}
\]
b) If Nyquist condition is satisfied, we would have $\hat{x}(t) = x(t)$. So, $\hat{x}(t)$ will be periodic since $x(t)$ is periodic.

However, if there is aliasing, $\hat{x}(t)$ will not always be periodic. For instance, let

$$T_s = 1 \implies f_s = 1 \quad \left( \text{Note that for no aliasing, we should have } |f_{\text{max}}| \leq 0.5 \right)$$

Consider

$$x(t) = e^{j2\pi 0.23t} + e^{j2\pi 0.46t} + e^{j2\pi 0.69t}$$

You can verify that $x(t + \frac{1}{0.23}) = x(t)$ for all $t$, hence $x(t)$ is periodic with $T = \frac{1}{0.23}$. Also, $e^{j2\pi 0.23t}$ and $e^{j2\pi 0.69t}$ terms will not cause aliasing; but $e^{j2\pi 0.46t}$ will.

Recall that $e^{j2\pi 0.69t}$ and $e^{-j2\pi 0.31t}$ produce the same samples when $T_s = 1$; and $e^{-j2\pi 0.31t}$ does not cause aliasing. Hence; we would have

$$\hat{x}(t) = e^{j2\pi 0.23t} + e^{j2\pi 0.46t} - e^{-j2\pi 0.31t}$$

which is NOT a periodic signal. So, in general, $\hat{x}(t)$ need not be periodic.
Q3 For each of the following discrete-time systems, determine whether the system is (i) Linear (ii) Causal (iii) Time-Invariant and (iv) FIR. Fully justify your answers for full credit.

(a) \( y[n] = x[n] \cdot x[n+1] \)
(b) \( y[n] = 3x[n] + 2n \)
(c) \( y[n+1] = 3x[n+1] - x[n] \).

* For linearity; consider \( x_1[n] \rightarrow y_1[n] \)
  \( x_2[n] \rightarrow y_2[n] \)
  form \( \alpha x_1[n] + \beta x_2[n] = x_3[n] \rightarrow y_3[n] \)
  Check whether \( y_3[n] = \alpha y_1[n] + \beta y_2[n] \) for all \( \alpha, \beta, x_1[n], x_2[n], n \)

* For time invariance; consider \( x[n] \rightarrow y[n] \)
  form \( x[n-n_0] = x_1[n] \rightarrow y_1[n] \)
  Check whether \( y_1[n] = y[n-n_0] \) for all \( x[n], n, n_0 \).

* For causality;
  check whether you need to know the values of \( x[n+1], x[n+2] \) -- to compute \( y[n] \) -- do this for all \( n \).

* For FIR;
  check whether the impulse response has a finite number of nonzero coefficients or not.

(a) \( \text{Linearity: } y_1[n] = x_1[n] \cdot x_1[n+1] \quad y_2[n] = x_2[n] \cdot x_2[n+1] \)
\( y_3[n] = x_3[n] \cdot x_3[n+1] = (\alpha x_1[n] + \beta x_2[n]) (\alpha x_1[n+1] + \beta x_2[n+1]) \)
\( = \alpha^2 y_1[n] + \beta^2 y_2[n] + \alpha \beta x_1[n] x_2[n+1] + \alpha \beta x_2[n] x_1[n+1] \neq \alpha y_1[n] + \beta y_2[n] \)

\( \Rightarrow \text{NOT linear. } \)
Time-Invariance:

\[ y_1[n] = x_1[n] x_1[n+1] \]
\[ = x_1[n-n_0] x_1[n-n_0+1] \]

Since \( y[n] = x[n] x[n+1] \), we see that
\[ y[n-n_0] = x[n-n_0] x[n-n_0+1] \]
(Replace \( n \) with \( n-n_0 \)).

\[ \Rightarrow y_1[n] = y[n-n_0] \Rightarrow \text{TIME - INVARIANT} \]

Causality:

To compute \( y[5] \), we need to know the value of \( x[6] \).
\[ \Rightarrow \text{NOT CAUSAL} \]

FIR:

\[ x[n] = 8[n] \Rightarrow y[n] = 0 \Rightarrow \text{FIR}. \]

(b) Linearity:

\[ y_1[n] = 3x_1[n] + 2n \]
\[ y_2[n] = 3x_2[n] + 2n \]
\[ y_3[n] = 3x_3[n] + 2n = 3x_1[n] + 3x_2[n] + 2n \]
\[ = 3y_1[n] + 3y_2[n] + 2n (1-\alpha-\beta) \]
\[ \neq \alpha y_1[n] + \beta y_2[n] \]

\[ \Rightarrow \text{NOT LINEAR} \]

Time-Invariance:

\[ y[n] = 3x[n] + 2n = 3x[n-n_0] + 2n \]
but \[ y[n-n_0] = 3x[n-n_0] + 2(n-n_0) \]
(Replace \( n \) with \( n-n_0 \)).

Since \( y_1[n] \neq y[n-n_0] \)
\[ \Rightarrow \text{NOT TIME - INVARIANT} \]

Causality:

Yes. To compute \( y[n] \) for any \( n_0 \), we just need to know \( x[n] \) at \( n=n_0 \).

FIR: NO. When \( x[n] = 8[n] \); \( y[n] = 38[n] + 2n \), which is not finite due to the \( 2n \) term.
First note that on both sides of
\[ y[n+1] = 3x[n+1] - x[n] \]
we can replace \( n \) with \( n-1 \) and nothing will change. Thus:
\[ y[n] = 3x[n] - x[n-1] \]
exactly represents the same system.

You can easily show that this system is a linear, time-invariant FIR system.

The system is also causal; because to compute \( y[n_0] \) for any \( n_0 \), we need to know \( x[n] \) only for \( n = n_0 \) and \( n = n_0 - 1 \). Future values (\( n = n_0 + 1 \), \( n_0 + 2 \), etc.) are not needed.
Q4. Given \( x(t) = \cos(6\pi t) + 2\sin(10\pi t) \), where \( t \) in seconds.

(a) Plot spectrum of \( x(t) \),

(b) Find the sampled signal \( x[n] = x(nT_s) \) where \( T_s = \frac{1}{16} \) sec.

(c) Plot the spectrum of \( x[n] \).

Fully justify your answers for full credit.

\[
\begin{align*}
\text{a) } x(t) &= \frac{1}{2} e^{j2\pi 3t} + \frac{1}{2} e^{-j2\pi 3t} + \frac{1}{3} e^{j\frac{2\pi 5}{16} t} - \frac{1}{3} e^{-j\frac{2\pi 5}{16} t} \\
\text{b) } x[n] &= \frac{1}{2} e^{j\frac{2\pi 3n}{16}} + \frac{1}{2} e^{-j\frac{2\pi 3n}{16}} + \frac{1}{3} e^{j\frac{2\pi 5n}{16}} - \frac{1}{3} e^{-j\frac{2\pi 5n}{16}} \\
\text{c) } x[n] &= \cos\left(\frac{6\pi n}{16}\right) + 2\sin\left(\frac{10\pi n}{16}\right)
\end{align*}
\]
(13pt) Q5. \( x[n] \rightarrow h[n] \rightarrow y[n] \)

Find the output \( y[n] \), for the following input \( x[n] \) and the impulse response \( h[n] \) of the above LTI system:

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] \] for all \( n \).

Since the system is LTI, we know that

\[ y[n] = 2x[n] + x[n-2] \]

In our case, \( h[n] = 2\delta[n] + \delta[n-2] \)

\[ \Rightarrow y[n] = 2x[n] + x[n-2] \]

Since \( x[n] = \delta[n+1] + \delta[n-1] - \delta[n-3] \), we get

\[ \Rightarrow y[n] = 2\delta[n+1] + 2\delta[n-1] - 2\delta[n-3] \]
\[ + \delta[n-1] + \delta[n-3] - \delta[n-5] \]

\[ \Rightarrow y[n] = 2\delta[n+1] + 3\delta[n-1] - 2\delta[n-3] - \delta[n-5] \]