

Derivation of Erlang B formula:

Consider a system with C channels and U users. Then

$A_u = H\lambda_1$ erlangs, where λ_1 is average call arrival rate for one user, and

$A = UA_u = H\lambda$:- $\lambda = U\lambda_1$, the average call arrival rate in the system.

1. Call arrivals is a random process in time and it is modelled as a *Poisson process*.

A random variable x is called Poisson distributed with parameter α if x takes on values $0, 1, 2, \dots, n, \dots$ with a probability

$$P\{x = k\} = e^{-\alpha} \frac{\alpha^k}{k!}$$

Hence the probability density function is

$$f(x) = e^{-\alpha} \sum_{k=0}^{\infty} \left(\frac{\alpha^k}{k!} \right) \delta(x-k)$$

with $p_k = P\{x=k\}$, and

$$p_k / p_{k+1} = (k+1)/\alpha.$$

p_k reaches to maximum at $k = [\alpha]$;

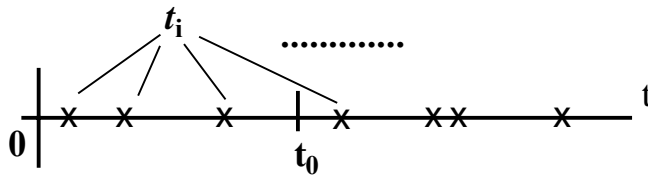
if $\alpha < 1$, p_k is max at $k=0$,

if $\alpha > 1$ and α is not an integer, p_k is max at $k = [\alpha]$,

if $\alpha > 1$ and α is an integer, p_k is max at $k = \alpha$ and $k = \alpha - 1$.

Poisson points:

Instants of call arrivals t_i are a set of points on t-axis:



Given a t_0 we define RV $a(t_0)$ as

$a(t_0)$ = total number of arrival instants t_i in interval $(0, t_0)$ with

$$P\{a(t_0) = k\} = \exp(-\lambda t_0) [(\lambda t_0)^k / k!].$$

Hence, number of points in an interval of length t_0 is also a Poisson distributed RV with parameter $\alpha = \lambda t_0$, where λ is the density of points (call arrivals; λ units: e.g. call arrivals/hr).

If the random process $a(t)$ is the total number of call arrivals until time t , then, the probability of having n new arrivals in $(t, t+\tau)$ is

$$P\{a(t+\tau) - a(t) = n\} = e^{-\lambda\tau} [(\lambda\tau)^n / n!]$$

where τ is a time interval.

2. User service time (H): Average duration of calls (or *average holding time*).

Service times are modelled by a RV which is exponentially distributed with mean duration of H. The probability that the service time s_n of n^{th} user is less than some call duration s is modelled as

$$\begin{aligned} P\{s_n < s\} &= 1 - \exp(-\mu s) && \text{for } s > 0 \\ &\approx \mu s && \text{if } \mu s \ll 1. \end{aligned}$$

with $\mu = 1/H$, the *mean service rate*, and

$$f(s) = \mu \exp(-\mu s).$$

The duration of every call is independent from the other. Therefore

$$\begin{aligned} P\{\text{the duration of at least one call, out of } i \text{ current calls in the system, is less than } s\} &= P\{s_1 < s\} + P\{s_2 < s\} + \dots + P\{s_i < s\} \\ &\approx i\mu s && \text{for small } \mu s. \end{aligned}$$

3. A trunked system:

The operation of a trunked system is a *continuous time – discrete state* random process:

Continuous-time: call can arrive at any time

Discrete-state: there can be discrete number of users;
state of the system *at any time instant* is the number of current users talking in the system.

This process can be modelled (an approximation) as a continuous time-discrete state *Markov* process.

We can sample this system at every δ seconds by sampling the number of current users. If we take these samples as the state the system, then the system is discretized on time axis also. Operation becomes:

Discrete time- discrete state Markov process.

A further simplification in modelling this operation is possible when δ is decreased. Assume that we keep δ sufficiently small such that, only a single one of the following events can take place within this interval:

- A current call can finish, i.e. $N_k = i, N_{k+1} = i-1$;
- Nothing happens, i.e. $N_k = i, N_{k+1} = i$;
- A new call arrives, i.e. $N_k = i, N_{k+1} = i+1$.

The next state can be at most one user more or one user less

Or, put in more rigorous terms,
the probability that

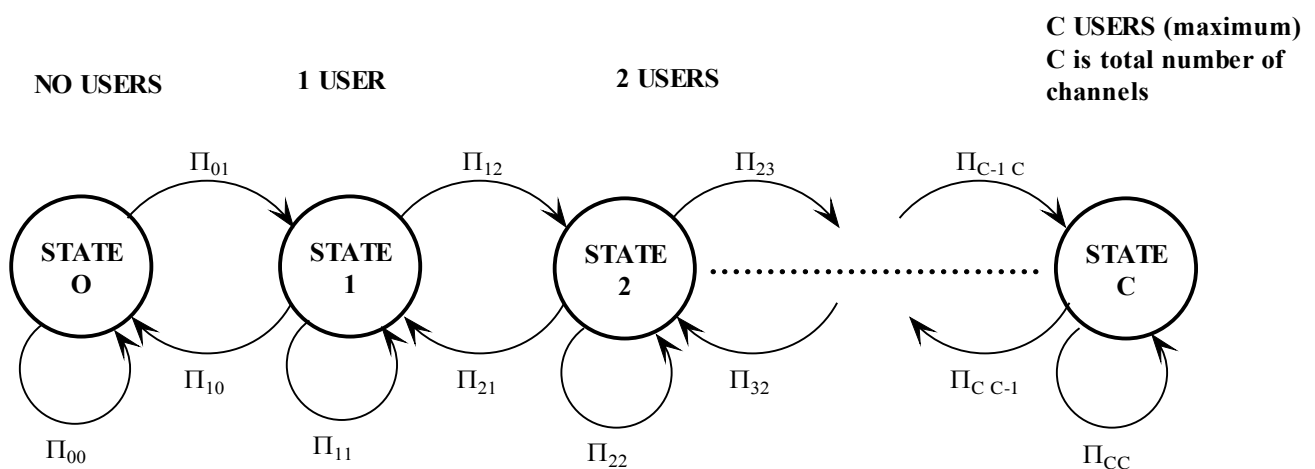
$1 - P\{\text{next state can be at most one user more or one user less}\}$
is insignificant.

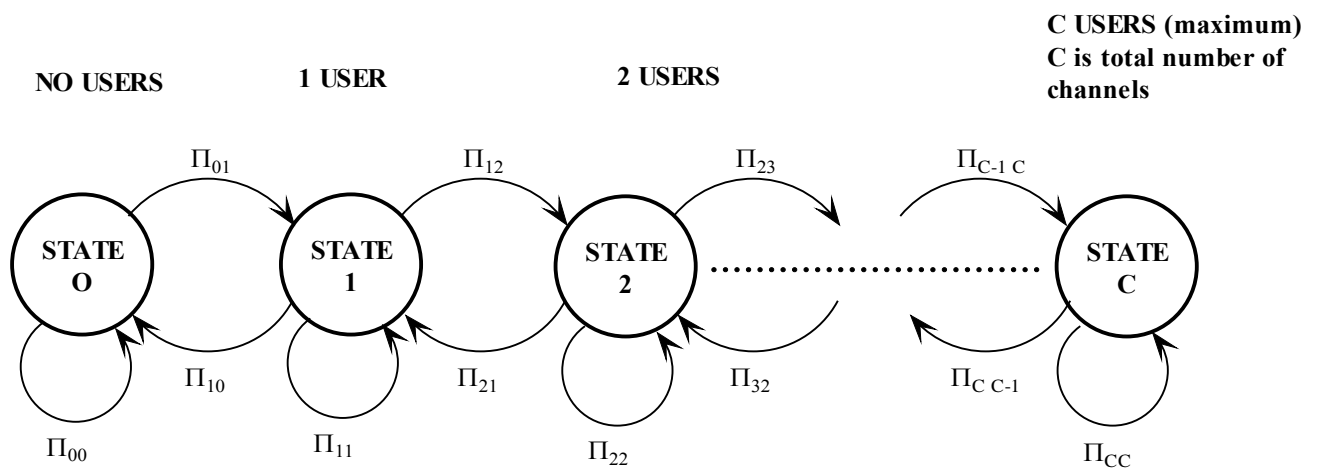
Now, let

$N_k = N(k\delta)$ is the number of occupied channels in the system (or the number of users in the system) at $t = k\delta$,

N_{k+1} can only be equal to $N_k - 1, N_k, N_k + 1$.

Graphically,





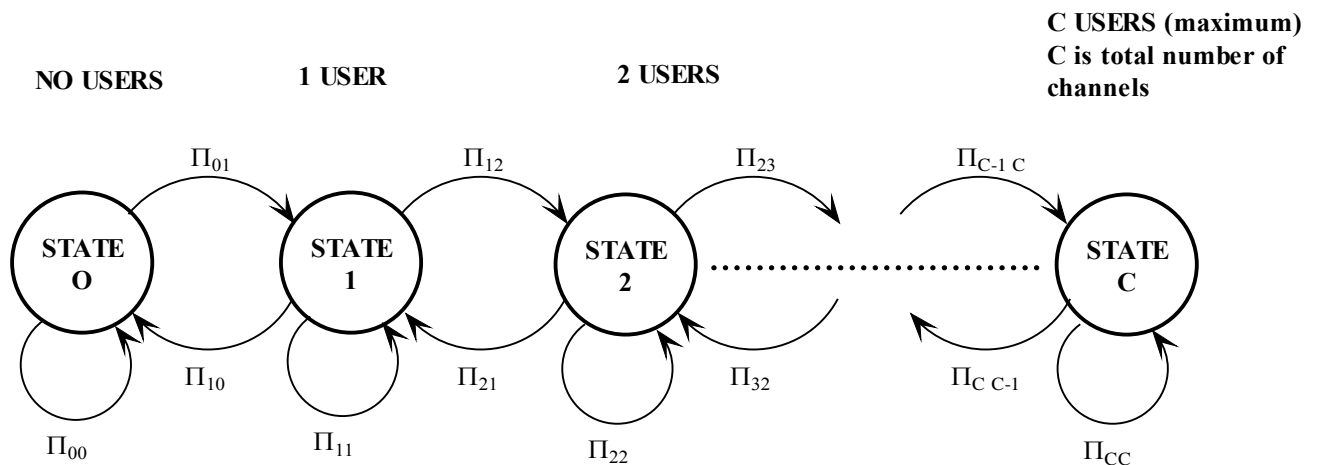
At any instant, state of the system can only be in any one of the nodes (circles). Π_{ij} are called the *transition probabilities* of the process, where $i, j = 0, 1, 2, \dots, C$.

A Markov process, in most general sense, (continuous-time/continuous-state) is a stochastic process whose past has no influence on the future, if its present state is specified.

A discrete-time *Markov chain* is a discrete Markov process which has *countable number of states*, as in this case. A Markov chain is specified in terms of its

(1) *state probabilities*: $P\{\text{process is at state } j \text{ at time instant } k\}$, or $P\{\text{there are } j \text{ users in the system at } t = k\delta\}$

$$p_j[k] = P\{N_k = j\} \quad j = 0, 1, 2, \dots, C \text{ (the value of the state)}$$



and its

(2) transition probabilities

$P\{\text{process is at state } j \text{ at time instant } k \text{ given that it was at state } i \text{ at time instant } l\},$

or

$P\{\text{there are } j \text{ users in the system at } t = k\delta \text{ given that there were } i \text{ users at time instant } l\}$

$$\Pi_{ij}[k,l] = P\{N_k = j \mid N_l = i\} \quad i, j = 0, 1, 2, \dots, C \text{ (values of states } N_l \text{ and } N_k) \text{ and } k, l \text{ are time instants } t = k\delta \text{ and } t = l\delta.$$

$\Pi_{ij}[k,l] = P\{N_k = j \mid N_l = i\}$ is the probability of transition from state i to another state.

If we sum these up over all states (over $j=0, 1, \dots, i, \dots, C$)

$$\sum_j \Pi_{ij}[k,l] = 1, \quad \text{since the process can be in one of } C \text{ states, anyway.}$$

Also, since

$$P\{N_k=j \mid N_l=i\} P\{N_l=i\} = P\{N_k=j, N_l=i\}, \quad (\text{remember?})$$

and

$$p_i[l] = P\{N_l=i\} \quad \text{state probability:} \\ P\{\text{there are } i \text{ users in the system at } t = l\delta\},$$

$$\begin{aligned} \sum_i p_i[l] \Pi_{ij}[k,l] &= \sum_i P\{N_l=i\} P\{N_k=j \mid N_l=i\} \\ &= \sum_i P\{N_k=j, N_l=i\} \\ &= P\{N_k=j\} \\ &= p_j[k] \quad (\text{again recall from joint probabilities!}) \end{aligned}$$

We must determine the transition probabilities. Recall that

$$P\{a(t+\delta) - a(t)=n\} = e^{-\lambda\delta} [(\lambda\delta)^n/n!],$$

and

$$P\{s_n < \delta\} = 1 - e^{-\mu\delta}. \text{ Then}$$

$$\Pi_{00}[k+1, k] = P\{\text{there are no call arrivals during time interval } [k\delta, (k+1)\delta], \\ \text{given that the state was } N_k=0 \text{ at } t = k\delta\}$$

is simply equal to the probability of having no call arrivals in $[k\delta, (k+1)\delta]$, or

$$\begin{aligned} \Pi_{00}[k+1, k] &= e^{-\lambda\delta} [(\lambda\delta)^0/0!] = e^{-\lambda\delta} \\ &\approx 1 - \lambda\delta \quad \text{for small } \lambda\delta. \end{aligned}$$

We shall denote $\Pi_{00}[k+1, k]$ by Π_{00} only from here onwards.

Similarly,

$$\begin{aligned}
 \Pi_{01} &= \Pi_{01}[k+1, k] = P\{N_{k+1}=1 \mid N_k=0\} \\
 &= P\{\text{state was 0 at } t=k\delta \text{ and one call arrived in } [k\delta, (k+1)\delta]\} \\
 &= e^{-\lambda\delta} [(\lambda\delta)^1/1!] \\
 &= (\lambda\delta)e^{-\lambda\delta} \approx (\lambda\delta)(1-\lambda\delta) \\
 &\approx \lambda\delta \quad \text{for small } \lambda\delta.
 \end{aligned}$$

$$\begin{aligned}
 \Pi_{10} &= \Pi_{10}[k+1, k] = P\{N_{k+1}=0 \mid N_k=1\} \\
 &= P\{\text{state was 1 and that call finished in } [k\delta, (k+1)\delta]\} \\
 &= 1 - e^{-\mu\delta} \\
 &\approx \mu\delta \quad \text{for } \mu\delta \ll 1.
 \end{aligned}$$

Similarly, around state i ,

$$\Pi_{i,i+1} = P\{N_{k+1}=i+1 \mid N_k=i\} \approx \lambda\delta, \quad \text{(one call arrived)}$$

and

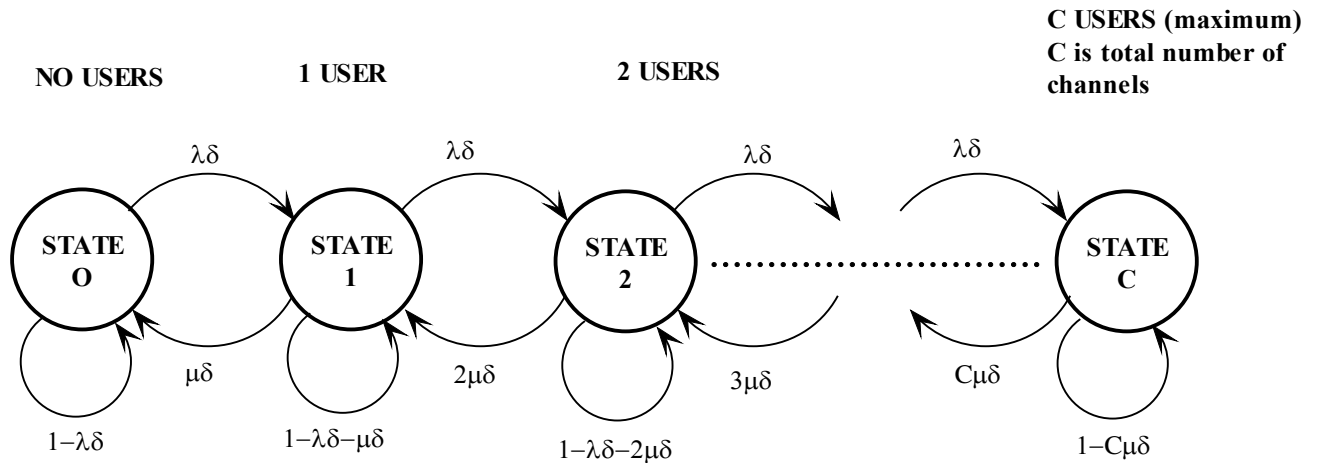
$$\begin{aligned}
 \Pi_{i,i-1} &= P\{N_{k+1}=i-1 \mid N_k=i\} = P\{\text{one call out of } i \text{ calls finished in } [k\delta, (k+1)\delta]\} \\
 &= P\{s_1 < \delta\} + P\{s_2 < \delta\} + \dots + P\{s_i < \delta\} \\
 &= i(1 - e^{-\mu\delta}) \\
 &\approx i\mu\delta \quad \text{for } \mu\delta \ll 1, \text{ since the events are independent.}
 \end{aligned}$$

And $\Pi_{i,i}$ {probability that the state remains at i },

We know that the total out-going probability from a node must sum up to unity in a Markov chain:

$$\begin{aligned}
 \Pi_{i,i+1} + \Pi_{i,i} + \Pi_{i,i-1} &= 1 \Rightarrow \Pi_{i,i} = 1 - \Pi_{i,i+1} - \Pi_{i,i-1} \\
 &= 1 - \lambda\delta - i\mu\delta.
 \end{aligned}$$

We can now redraw the Markov chain as



But a Markov process is also Markov if the time is reversed:

$$P\{N_{k+1}|N_k\}P\{N_k\} = P\{N_{k+1}, N_k\} = P\{N_k|N_{k+1}\}P\{N_{k+1}\}.$$

Now, for any $n \leq C$

$$\begin{aligned} P\{N_k=n-1\}P\{N_{k+1}=n|N_k=n-1\} &= P\{N_k=n-1\}(\lambda\delta) \\ &= P\{N_{k+1}=n\}P\{N_k=n-1|N_{k+1}=n\} = P\{N_{k+1}=n\}(n\mu\delta) \end{aligned}$$

$$\Rightarrow P\{N_k=n-1\}(\lambda\delta) = P\{N_{k+1}=n\}(n\mu\delta), \text{ or}$$

$$P_{n-1}(\lambda\delta) = P_n(n\mu\delta)$$

$$\Rightarrow \lambda P_{n-1} = n\mu P_n.$$

Thus

$$\begin{aligned} P_1 &= (\lambda/\mu) P_0 \\ P_2 &= (\lambda/2\mu) P_1 = (1/2)(\lambda/\mu)^2 P_0 \\ &\cdot \\ &\cdot \\ &\cdot \\ P_n &= (\lambda/n\mu) P_{n-1} = (\lambda/\mu)^n P_0 / n! \end{aligned}$$

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&\cdot \\
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\end{aligned}$$

We also know that

$$\sum_{i=0}^C P_i = 1 .$$

Substituting P_i in Σ :

$$\sum_{i=0}^C (\lambda/\mu)^i P_0 / i! = 1 \quad \Rightarrow \quad P_0 = \left\{ \sum_{i=0}^C (\lambda/\mu)^i / i! \right\}^{-1}$$

Probability of having all channels occupied is:

$$P_C = (\lambda/\mu)^C P_0 / C! = \{ (\lambda/\mu)^C / C! \} / \left\{ \sum_{i=0}^C (\lambda/\mu)^i / i! \right\}$$

The total offered traffic is $A = \lambda H = \lambda/\mu$ Erlangs. Then

$$P_C = \{ (A)^C / C! \} / \left\{ \sum_{i=0}^C (A)^i / i! \right\} = \text{GOS} = \text{Probability of blocking}$$

The Erlang B formula!