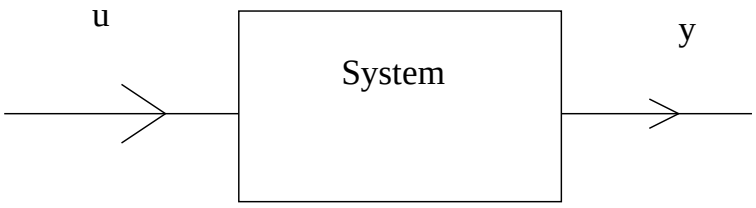
Nonlinear Systems : Chapter 1

Linear vs. Nonlinear



- $u_1 \rightarrow y_1 \qquad , \qquad u_2 \rightarrow y_2$
- If we have $\alpha_1 u_1 + \alpha_2 u_2 \rightarrow \alpha_1 y_1 + \alpha_2 y_2$
- $\forall \;\; \alpha_1, \; \alpha_2, \; u_1, \; u_2 \Longrightarrow$ System is **Linear**
- otherwise $\Longrightarrow$ System is **Nonlinear**
- If the system is dynamic $\Longrightarrow$ Zero initial conditions
- Simple examples :
- $y = 2u \;\; \Longrightarrow$ **Linear**
- $y = u^2 \;\; \Longrightarrow$ **Nonlinear**

- **General Dynamical Systems :**

- State Equations $\begin{cases} \dot{x}_1 = f_1(t, x_1, \ldots . x_n, u_1, \ldots, u_m) \\ \vdots \\ \dot{x}_n = f_n(t, x_1, \ldots . x_n, u_1, \ldots, u_m) \end{cases}$

- Outputs $\begin{cases} y_1 = h_1(t, x_1, \ldots . x_n, u_1, \ldots, u_m) \\ \vdots \\ y_p = h_p(t, x_1, \ldots . x_n, u_1, \ldots, u_m) \end{cases}$

- $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{R^n} \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbf{R^m} \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \in \mathbf{R^p}$

- $f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in \mathbf{R^n} \qquad h = \begin{pmatrix} h_1 \\ \vdots \\ h_p \end{pmatrix} \in \mathbf{R^p}$

- System Equations $\begin{cases} \dot{x} = f(t, x, u) \\ y = h(t, x, u) \end{cases}$

- Linear System Equations $\begin{cases} \dot{x} = A(t)x + B(t)u \\ y = C(t)x + D(t)u \end{cases}$

- This is **Time Varying** Case

- Linear Time Invariant Case $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

- **Aim :**
- **i :** Given u, find (or predict) y and/or $x \Longrightarrow$ **Analysis**
- **ii :** Given a **desired behaviour**, find an **appropriate** u
- $\Longrightarrow$ **Design**

- In some cases, the system may be given as follows :
- $y^{(n)} = f(t, y, \dot{y}, \ddot{y}, \ldots, y^{(n-1)}, u)$
- **Standard trick :** $x_1 = y,\ x_2 = \dot{y},\ x_3 = \ddot{y}, \ldots, x_n = y^{(n-1)}$
- $\Downarrow$
- $$\left.\begin{array}{c} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = f(t, x_1, \ldots, x_n, u) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \dot{x} = F(t, x, u) \\ y = x_1 = Cx = (1\ 0\ \ldots\ 0)x \end{array}\right.$$

- A special case : Systems **affine** in input
- $$\Longrightarrow \left\{\begin{array}{c} \dot{x} = F(x) + G(x)u \\ y = H(x) \end{array}\right.$$

• **Some concepts and questions related to Nonlinear Systems**

• Consider the ODE $\dot{x} = f(t, x)$

• **Concept of a solution** $x(t)$: Given an initial time t_0, and initial condition $x(t_0) = x_0$, $x(t)$ is called a **solution** of ODE if :

• **i :** $x(t_0) = x_0$

• **ii:** for some $T > t_0$, we have $\dot{x}(t) = f(t, x(t)) \qquad t_0 \geq t < T$

• **4 questions related with solutions :**

• **i :** At least one solution exists (**existence**)

• **ii :** Exactly one solution exists for $T < \infty$ (**local uniqueness**)

• **iii :** Case **ii** + $T = \infty$ (**global uniqueness**)

• **iv :** Case **iii** + solution depends continuously on initial condition (**well-posedness**)

• **Linear Case :** $\dot{x} = Ax \Longrightarrow x(t) = e^{A(t-t_0)}x(t_0)$

• $e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots + \frac{(At)^n}{n!} + \ldots$

• **All 4 conditions are satisfied !**

• **Linear Case with input** : $\dot{x} = Ax + Bu$

• $\Longrightarrow \qquad x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$

• $\qquad\qquad\qquad\quad \Downarrow \qquad\qquad\qquad\quad \Downarrow$

• $\Longrightarrow \qquad x(t) =$ Zero-Input Solution + Zero-State Solution

- For Nonlinear case, these 4 questions has no positive answer, in general

- For existence :

- $\dot{x} = -\text{sign}x \Longrightarrow \text{sign}x = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

- For $x(0) = 0$, no solution exists

- For $\dot{x} = \text{sign}x$, solutions exist.

- For Uniqueness :

- $\dot{x} = 3x^{2/3}$

- $x(0) = 0, \quad \forall \alpha > 0 \quad \Longrightarrow x(t) = \begin{cases} 0 & t < \alpha \\ (t-\alpha)^3 & t \geq \alpha \end{cases}$

- Infinitely many solutions !

- Consider the ODE $\dot{x} = f(t, x)$

- If f depends on time $\Longrightarrow$ **non-autonomous system**

- If f does not depend on time $\Longrightarrow$ **autonomous system**

- x_0 is called an **equilibrium point** if $f(t, x_0) = 0 \quad \forall\ t \geq 0$

- Importance : If $x(t_0) = x_0 \quad \Longrightarrow \quad x(t) = x_0 \qquad \forall\ t \geq t_0$

- Importance : Same as **DC Operating Point**

- Importance : All Linearizations are usually made around equilibrium points

- $\Longrightarrow$ **Small Signal Behaviour**

- Linearization around equilibrium points
- Consider System Equations $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases} \Longrightarrow$ Time-Invariant
- Assume that we have $x = x^*,\ u = u^*$
- $\Longrightarrow f(x^*, u^*) = 0 \Longrightarrow$ Equilibrium
- $\dot{x} = f(x^*,u^*) + \frac{\partial f}{\partial x}|_{x=x^*,u=u^*}(x-x^*) + \frac{\partial f}{\partial u}|_{x=x^*,u=u^*}(u-u^*) + \text{h.o.t}$

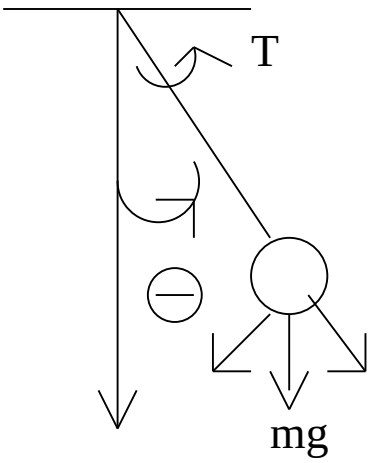
- Define $\delta x = x - x^*,\ \delta u = u - u^*,\ A = \frac{\partial f}{\partial x}|_{x=x^*,u=u^*},\ B = \frac{\partial f}{\partial u}|_{x=x^*,u=u^*}$
- $\dot{\delta x} = \dot{x} = A\delta x + B\delta u$
- Can do the same for the outputs
- $y = h(x^*,u^*) + \frac{\partial h}{\partial x}|_{x=x^*,u=u^*}(x-x^*) + \frac{\partial h}{\partial u}|_{x=x^*,u=u^*}(u-u^*) + \text{h.o.t}$

- Define $y^* = h(x^*,u^*),\ \ \delta y = y - y^*,\ C = \frac{\partial h}{\partial x}|_{x=x^*,u=u^*},\ D = \frac{\partial h}{\partial u}|_{x=x^*,u=u^*}$
- $\delta y = C\delta x + D\delta u$
- $\delta x,\ \delta y \Longrightarrow$ **Small Signal Solutions**, $\delta u \Longrightarrow$ **Small Signal Input**
- Example : $\quad \dot{x} = -x - x^3 + u - 1 \qquad y = \sin x + u^2$
- Equilibrium point : $x^* = 0,\ u^* = 1$
- $a = -1,\ b = 1,\ c = 1,\ , d = 2$
- $\dot{\delta x} = -\delta x + \delta u$
- $\delta y = \delta x + 2\delta u$

- **Superposition does not apply in nonlinear case**

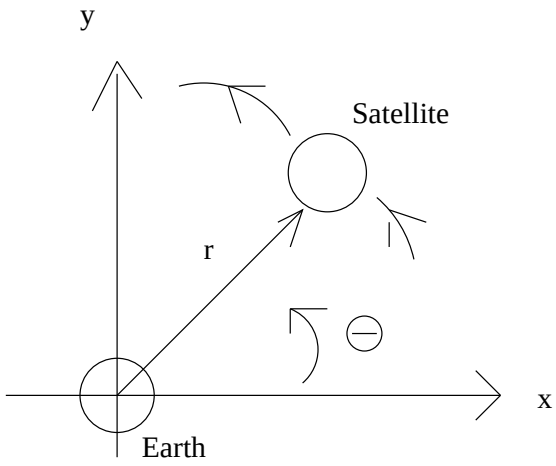
• **Sources of Nonlinearity :**

- **i :** Physical laws may impose nonlinearity
- Example : Pendulum (One-link robot arm)



- Physical Law : $\frac{d}{dt}$ (Angular Momentum) = Applied Torque
- Physical Law : Conservation of angular momentum
- $I\ddot{\theta} + mgl \sin\theta = \tau$
- For linearization : $\sin\theta \simeq \theta$ for small angles
- $\Longrightarrow I\ddot{\theta} + mgl\theta = \tau$
- If $\tau = 0$, $\Longrightarrow$ small signal solutions are oscillatory.

- Example : Satellite in Orbit



- Physical Law : $\frac{d}{dt}$ (Linear Momentum) = Applied Force $\qquad (F = ma)$

- $\vec{r} \;=\; r\cos\theta \;\; \vec{e}_x + r\sin\theta \;\; \vec{e}_y$

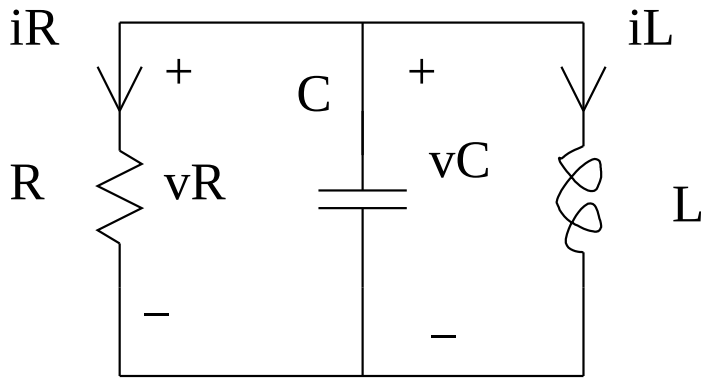
- $\frac{d^2}{dt^2}\;\vec{r} = F - \frac{kmM}{|\;\vec{r}\;|^3} \;\; \vec{r}$

- $\frac{d^2}{dt^2}\;\vec{r} = (\ddot{r}\cos\theta - 2r\sin\theta\dot{\theta} - \dot{r}\sin\theta\dot{\theta} - r\cos\theta\dot{\theta}^2)\;\vec{e}_x$
- $\qquad\qquad +(\ddot{r}\sin\theta + 2r\cos\theta\dot{\theta} + \dot{r}\cos\theta\dot{\theta} - r\sin\theta\dot{\theta}^2)\;\vec{e}_y$

- Final equations are **highly nonlinear**

• **ii :** Physical laws may be linear, but element relations may be nonlinear

• Example : Nonlinear Circuits

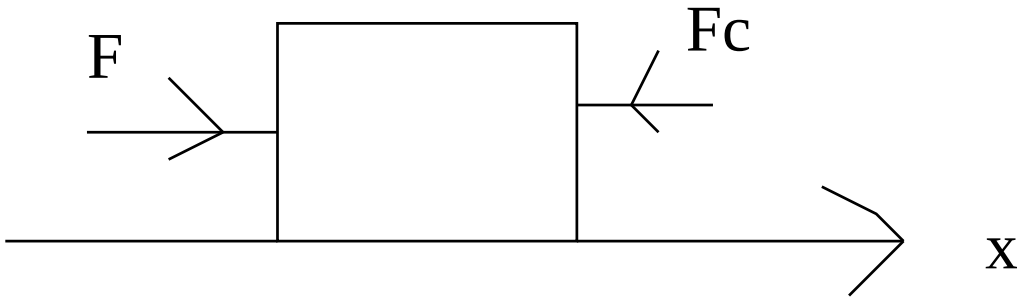


• $C\dot{v}_C + i_L + Gv_C = 0 \quad , \quad L\dot{i}_L = v_C$

• What if the resistor is **nonlinear**? i.e. $i_R = g(v_R)$

• $C\dot{v}_C + i_L + g(v_C) = 0 \quad , \quad L\dot{i}_L = v_C$

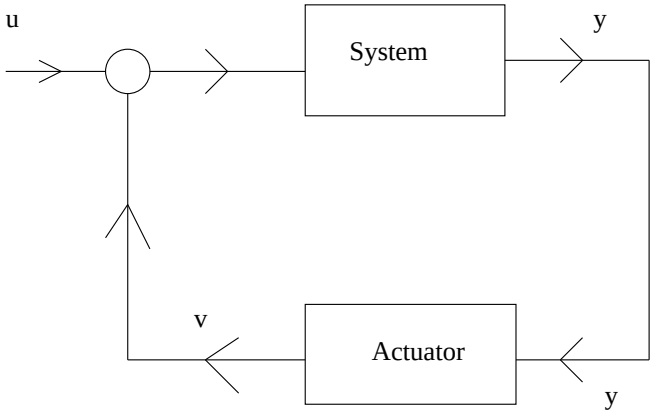
• Example : Dry Friction

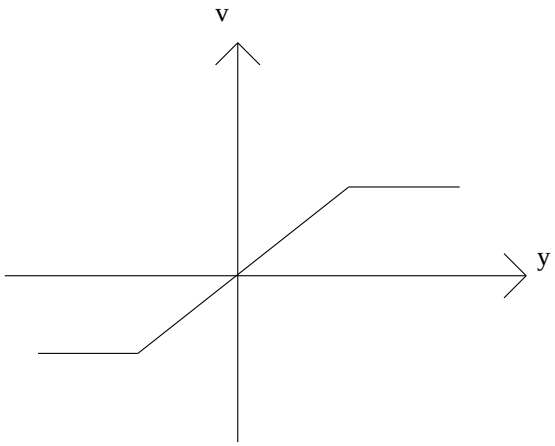


• $m\dot{x} = F + F_c \quad \Longrightarrow \quad F_c = -\mu_k mg \ \ \text{sign}(\dot{x})$

• $m\dot{x} + \mu_k mg \ \ \text{sign}(\dot{x}) = F$

- Example : Some elements may become nonlinear in some range
- typical examples in control systems : actuator saturation for large input signals
- This may cause instabilities, oscillations, etc.





- **Other Differences Between Linear and Nonlinear Systems**

- **Finite Escape Time**

- In linear systems, solutions exist $\forall t$
- In nonlinear systems, solutions may blow-up in finite time

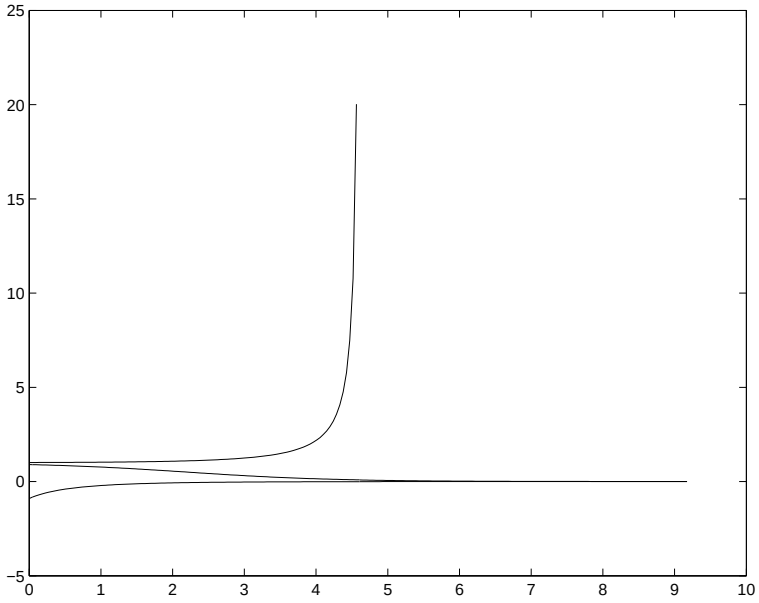
- Example : $\dot{x} = -x + x^2, \quad x(0) = x_0$

- Equilibrium points : $x_1 = 0, \; x_2 = 1$

- $\Longrightarrow x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$

- If $x_0 < 1$, $\Longrightarrow x(t) \to 0$ as $t \to \infty$
- If $x_0 > 1$, $\Longrightarrow x(t) \to \infty$ as $t \to T$ for some $T < \infty$

- $\qquad 1 - x_0 + x_0 e^{-T} = 0 \Longrightarrow T = \ln \frac{x_0}{x_0 - 1} \; > \; 0$



- This behaviour cannot be seen in Linear Systems

- Does linearization always give good solutions ?

- Example : $\dot{x} = kx^3 \Longrightarrow$ Eql. point : $x^* = 0$
- Linearization $\Longrightarrow \dot{\delta x} = 0 \Longrightarrow \delta x =$ constant

- How about the original system? Auxiliary function $V(x) = x^2$

- $\dot{V}(x(t)) = \frac{dV}{dx}\ \frac{dx}{dt} = 2x\dot{x} = 2kx^4$

- $k > 0 \implies \dot{V}(x(t)) > 0 \implies V \uparrow \implies |\, x(t)\,| \uparrow$

- $k < 0 \implies \dot{V}(x(t)) < 0 \implies V \downarrow \implies |\, x(t)\,| \downarrow$

- Conclusion : The behaviour of the linearized system and original system may be quite different

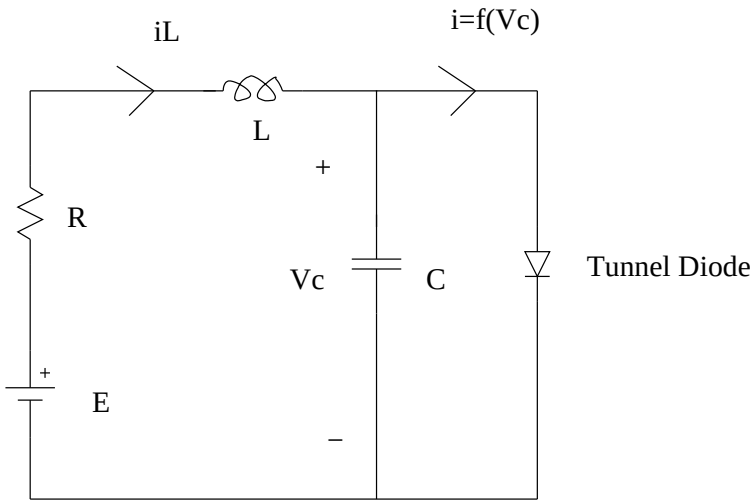
- We need tools to analyze the original system without actually finding the solution

- **Multiple Equilibrium points :**

- In linear case $\Longrightarrow \dot{x} = Ax \Longrightarrow Ax = 0$
- $\Longrightarrow$ either **unique** equilibrium point $x = 0$ or **infinitely many**

- In nonlinear case $\Longrightarrow \dot{x} = f(x) \Longrightarrow f(x) = 0$
- $\Longrightarrow$ there may be **finitely many** equilibrium points
- Example : $\dot{x} = -x + x^2 \Longrightarrow -x + x^2 = 0 \Longrightarrow x_1 = 0,\ x_2 = 1$

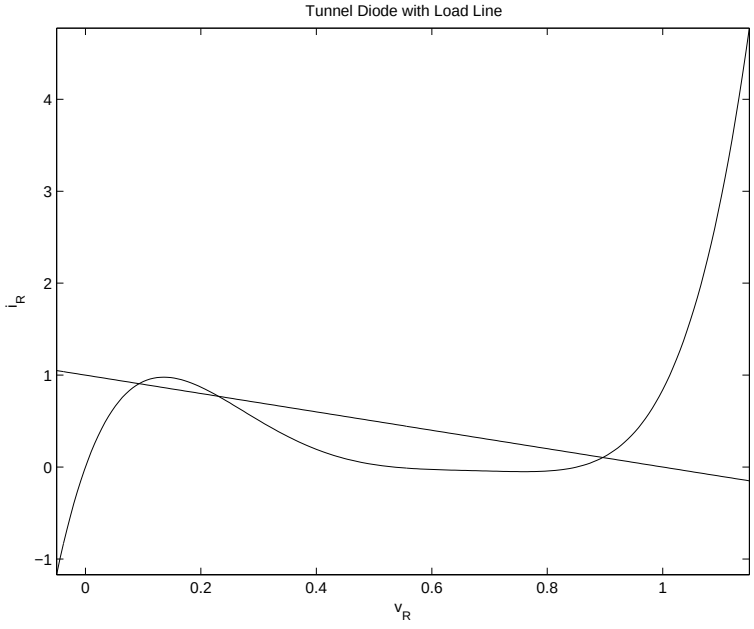
- Example : Tunnel Diode Circuit



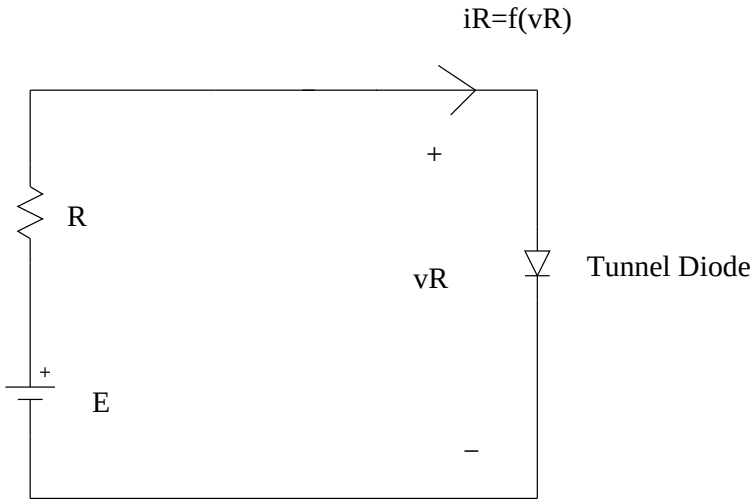
- $C\dot{v}_C = i_L - f(v_C) \quad , \quad L\dot{i}_L = -Ri_L - v_C + E$

- Equilibrium : $i_L - f(v_C) = 0 \ , \ -Ri_L - v_C + E = 0 \Longrightarrow E = v_C + Rf(v_C)$

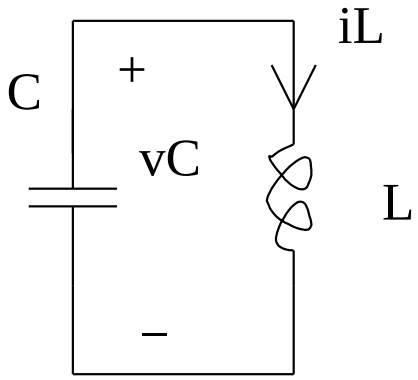
- Geometric Interpretation : $v_C = v_R \ , \ E = v_R + Ri_R \ , \ i_R = f(v_R)$



- Circuit Interpretation : $\dot{v_C} = 0 \Longrightarrow i_C = 0 \Longrightarrow$ Capacitor is **open**
- Circuit Interpretation : $\dot{i_L} = 0 \Longrightarrow v_L = 0 \Longrightarrow$ Inductor is **short**

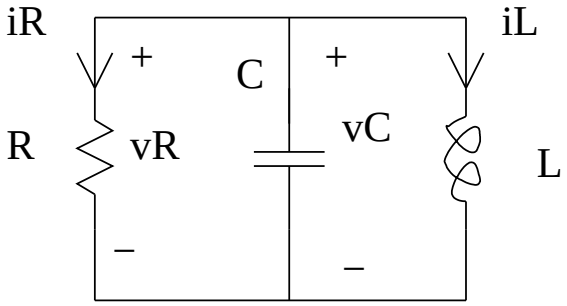


- **Limit Cycles :**
- Limit cycles are **isolated** oscilations
- Simple oscillator in Linear Case :



- $C\dot{v_C} = -i_L \quad , \quad L\dot{i_L} = v_C$
- This is also called **Harmonic Oscillator**

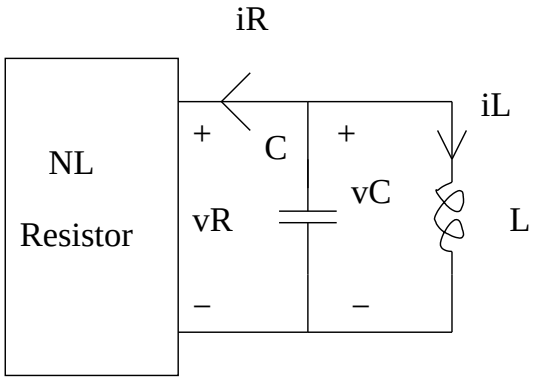
- $\Longrightarrow \ddot{v}_C = -\frac{1}{LC} v_C \quad , \quad \ddot{i}_L = -\frac{1}{LC} i_L$
- $v_C(t) = C_1 \cos(\omega t + \phi_C) \quad , \quad i_L(t) = C_2 \cos(\omega t + \phi_L) \quad , \quad \omega = 1/\sqrt{LC}$
- Problem 1 : Magnitudes depend on the **initial conditions**
- $\Longrightarrow$ Oscillations are **not isolated**
- Energy : $E(t) = 0.5 L i_L^2 + 0.5 C v_C^2$
- How does this energy evolve :
- $\dot{E} = L i_L \dot{i}_L + C v_C \dot{v}_C = i_L v_C + v_C(-i_L) = 0$
- $\Longrightarrow$ Energy is conserved ! $\Longrightarrow E(t) = E(0)$
- Problem 2 : These oscillations are not **structurally stable**
- Usually capacitor and/or inductor has a small resistance



- $C\dot{v}_C = -i_L - G v_C \quad , \quad L\dot{i}_L = v_C$

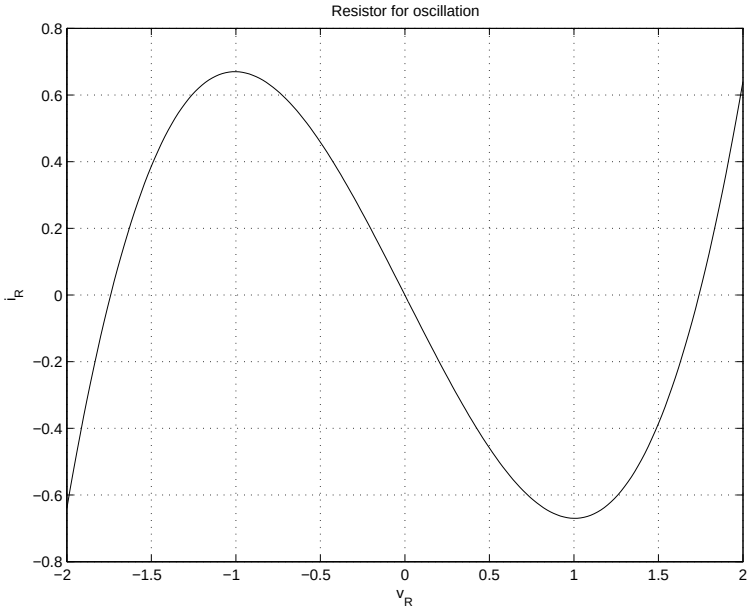
- Energy : $E(t) = 0.5Li_L^2 + 0.5Cv_C^2$
- How does this energy evolve :
- $\dot{E} = Li_L\dot{i}_L + Cv_C\dot{v}_C = i_Lv_C + v_C(-i_L - Gv_C) = -Gv_C^2 < 0$
- $\Longrightarrow$ Energy is **not** conserved ! $\Longrightarrow$ $E(t) \downarrow$ as long as $v_C \neq 0$
- $\Longrightarrow v_C = 0 \Longrightarrow \dot{v}_C = 0 \Longrightarrow i_L = 0$
- $\Longrightarrow$ LaSalle's Invariance Principle !

- How do we sustain oscillations, which does not depend on initial conditions ? $\Longrightarrow$ **Limit Cycles**
- Typical electronic oscillator :



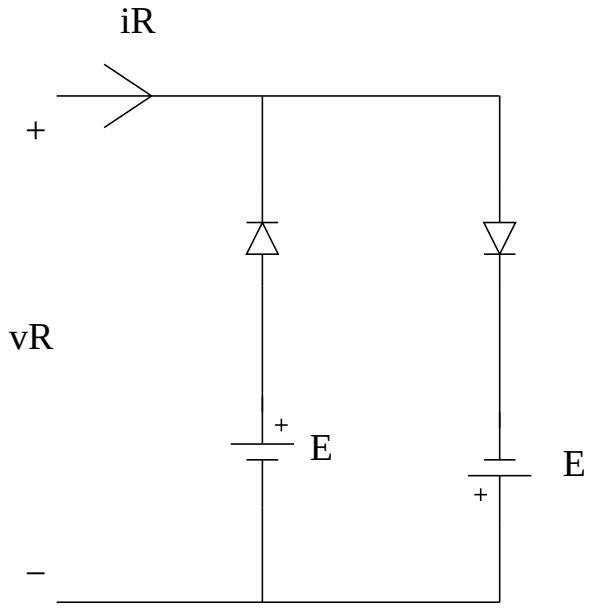
- Nonlinear resistor should be **negative** near origin, but **positive** away from origin

- Typical $i - v$ behaviour of nonlinear resistor :

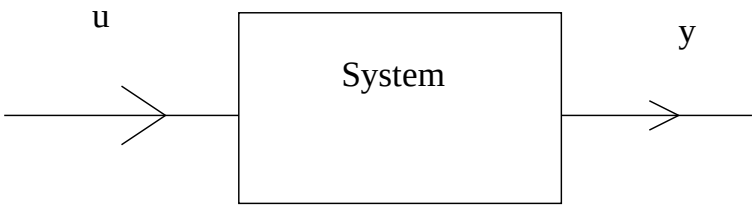


- Basic requirements : $f(0) = 0,\ f'(0) < 0,\ f(v) \to \pm\infty$ as $v \to \pm\infty$

- Typical Electronic Realization :



• **Subharmonic Generation :**



- In linear case, $u = A\cos(\omega t + \phi_i) \rightarrow y = B\cos(\omega t + \phi_o)$
- $\Longrightarrow u = \Sigma_j\, A_j \cos(\omega_j t + \phi_{ij}) \rightarrow y = \Sigma_j\, B_j \cos(\omega_j t + \phi_{oj})$

- In nonlinear case, $u = A\cos(\omega_1 t + \phi_i) \rightarrow y = B\cos(\omega_2 t + \phi_o)$
- where $\omega_1 \neq \omega_2$ may happen

- $\omega_2 < \omega_1 \Longrightarrow$ Subharmonic generation
- $\omega_2 = k\omega_1 \Longrightarrow$ where k is integer $\Longrightarrow$ harmonic generation

- Sometimes this is useful $\Longrightarrow$ **modulation**
- Sometimes this is not useful $\Longrightarrow$ **hi-fi audio system**

- Why do we need something more than linearization ?

- Example : $\dot{x} = x + x^2 + u \;\; . \;\; y = x$

- If $u = 0 \Longrightarrow$ eql. point $x = 0$, $x = -1$

- Linearization around $x = 0 \Longrightarrow \dot{\delta x} = \delta x$
- $\Longrightarrow \delta x = e^t \delta x(0) \Longrightarrow$ unstable

- Use feedback law $u = -2y = -2x \Longrightarrow \dot{x} = -x + x^2$

- $\Longrightarrow$ eql. point $x = 0$, $x = 1$
- Linearization around $x = 0 \Longrightarrow \dot{\delta x} = -\delta x$
- $\Longrightarrow \delta x = e^{-t}\delta x(0) \Longrightarrow$ **stable**

- However original system has **finite escape time** when $x(0) > 1$
- Linearization around $x = 1 \Longrightarrow f'(1) = 2x - 1 \mid_{x=1} = 1$
- $\Longrightarrow \dot{\delta x} = \delta x \Longrightarrow \delta x = e^{t}\delta x(0) \Longrightarrow$ unstable

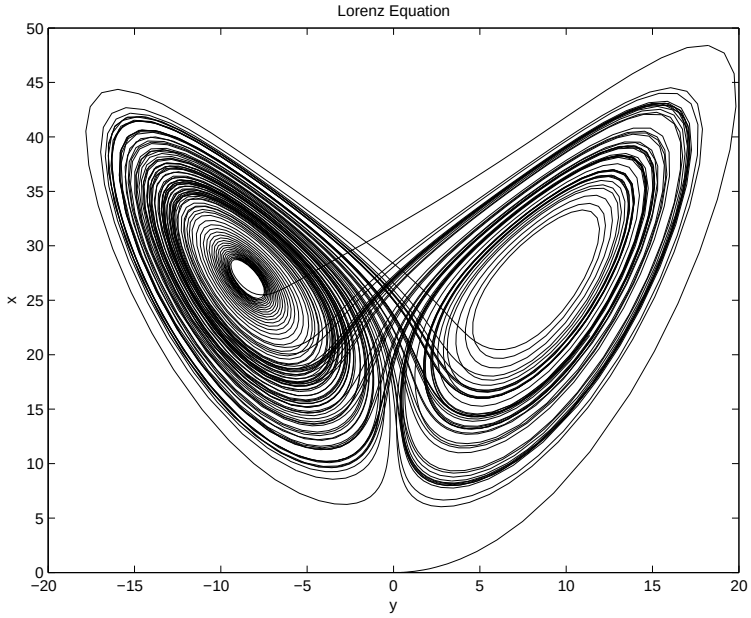
- **Chaotic Behaviour**

- In most of the systems, the solutions are either bounded or unbounded
- bounded solutions, either converge to a point, or oscillatory, or almost periodic
- In chaotic systems, $x_0 \rightarrow x(t),\ x_0 + \delta x(0) \rightarrow x(t) + \delta x(t)$
- we have $\delta x = e^{\lambda t}\delta x(0)$ for small time
- but the solutions still remain **bounded**

- $\Longrightarrow$ Sensitive dependence on initial conditions

- $\Longrightarrow$ Example : **Lorenz Equation**

- $\Longrightarrow \dot{x} = -bx + yz$
- $\Longrightarrow \dot{y} = sz - sy$
- $\Longrightarrow \dot{z} = -xy + ry - z$
- $\Longrightarrow s = 10;,\ r = 28,\ b = 8/3$



- $\Longrightarrow$ May model meteorologic behaviour
- $\Longrightarrow$ Sensitive dependence makes long time prediction almost **impossible**
- $\Longrightarrow$ Numerical simulations also pose problems

- **What do we need ?**

- We can't have closed-form solutions. So we need some **qualitative** analysis techniques, i.e. predicting the **behaviour** without actually solving the system.

- There is not a **single** technique applicable to all nonlinear systems. So we need different techniques.

- The mathematics is more complicated than the linear case