# POLYNOMIAL KERNELS

## EXOR FUNCTION.

$$x_1 \oplus x_2 = \begin{cases} 1 & x_1 \neq x_2 \\ 0 & x_1 = x_2 \end{cases}$$

| $x_1$ | $x_2$ | o |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

actually $x_1 \oplus x_2 = x_1 + x_2 \pmod 2$ $\Rightarrow$ parity check.

$x_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $x_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $x_C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $x_D = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow$ Choose $\{-1, 1\}$ representation (better for polynomials)

$x_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Downarrow d_1 = -1$ $x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Downarrow d_2 = 1$ $x_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Downarrow d_3 = 1$ $x_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Downarrow d_4 = -1$

NOT LINEARLY SEPARABLE $\Rightarrow$ Linear techniques do not work.

$$\underbrace{x = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^2}_{\text{not separable in 2D}} \longrightarrow \underbrace{\phi(x) = \begin{pmatrix} 1 \\ z_1^2 \\ \sqrt{2}\, z_1 z_2 \\ z_2^2 \\ \sqrt{2}\, z_1 \\ \sqrt{2}\, z_2 \end{pmatrix} \in \mathbb{R}^6}_{\text{possibly separable in 6D.}}$$

$$K(x_1, x_2) = \phi(x_1)^T \phi(x_2) = (1 + x_1^T x_2)^2$$

$$K(x_i, x_j) = \phi(x_i)^T \phi(x_j) \qquad i=1,2,3,4 \quad j=1,2,3,4$$

$$K = \begin{bmatrix} 9 & 1 & 1 & 1 \\ 1 & 9 & 1 & 1 \\ 1 & 1 & 9 & 1 \\ 1 & 1 & 1 & 9 \end{bmatrix}$$

$$L(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{i=1}^{4} \alpha_i - \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \alpha_i \alpha_j d_i d_j K(x_i, x_j)$$

$$\frac{\partial L}{\partial \alpha_i} = 0$$

$$\Rightarrow \boxed{\begin{aligned} 9\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 &= 1 \\ -\alpha_1 + 9\alpha_2 + \alpha_3 - \alpha_4 &= 1 \\ -\alpha_1 + \alpha_2 + 9\alpha_3 - \alpha_4 &= 1 \\ \alpha_1 - \alpha_2 - \alpha_3 + 9\alpha_4 &= 1 \end{aligned}}$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{8} \Rightarrow$ All 4 points are SUPPORT VECTORS.

$$w = \sum_{i=1}^{4} \alpha_i d_i \phi(x_i) = \begin{pmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\theta = w^T \phi(x_i) - d_i = 0$$

$$\boxed{g(x) = w^T \phi(x) - \theta = w^T \phi(x) = -\frac{1}{\sqrt{2}} \sqrt{2}\, x_1 x_2 = -x_1 x_2}$$

$g(x_i) > 0 \Leftrightarrow x_i \in C_1$, $g(x_i) < 0 \Rightarrow x_i \in C_2$.