PROBLEM: Given the training set $\{y_1, y_2, \dots, y_p\}$ of patterns, find W and θ such that

$$o(k+1) = \Gamma(W o(k) - \theta)$$

solves problems 1-4 of previous slide.

$$y_i = \begin{pmatrix} y_{i1} \\ \vdots \\ y_{in} \end{pmatrix} \in \mathbb{R}^n$$

OUTER PRODUCT RULE: Related to Hebbian Learning

$$W = \frac{1}{n}\left(y_1 y_1^T + y_2 y_2^T + \dots + y_p y_p^T\right) \qquad \theta = 0$$

$$= \frac{1}{n} \sum_{i=1}^{p} y_i y_i^T$$

$\Rightarrow W = W^T$! But $w_{ii} \neq 0$ in general.

Assume that y_i's are binary vectors ($\Rightarrow y_{ij} = \mp 1$)

$$y_i y_i^T = \begin{pmatrix} \mp 1 \\ \mp 1 \\ \vdots \\ \mp 1 \end{pmatrix} (\mp 1 \cdots \mp 1) = \begin{pmatrix} 1 & & * \\ & 1 & \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

$\Rightarrow$ diagonal entries of $\sum_{i=1}^{p} y_i y_i^T \Rightarrow p$!

$\Rightarrow$ To get $w_{ii} = 0$

$$\Rightarrow \quad \boxed{W = \frac{1}{n} \sum_{i=1}^{p} y_i y_i^T - \frac{p}{n} I} \qquad \theta = 0.$$

$\Rightarrow$ This choice satisfies conditions 3-4.

* Fixed point?

$$\boxed{o(k+1) = \Gamma(W o(k))} \quad k=0,1,\ldots$$

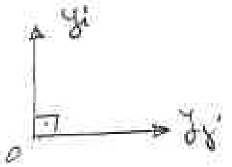
$o(0) = y_i$. do we have $o(1) = o(2) = \cdots = y_i$?

$$v(1) = W y_i = \left(\frac{1}{n} \sum_{j=1}^{p} y_j y_j^T - \frac{p}{n} I \right) y_i$$

$$= \frac{1}{n} \left(y_1 y_1^T y_i + \cdots + y_i y_i^T y_i + \cdots + y_p y_p^T y_i \right) - \frac{p}{n} y_i$$

How do we guarantee that $y_i = \Gamma(W y_i)$?

* Assume that patterns are ORTHOGONAL! $\boxed{y_i^T y_j = 0 \quad i \neq j}$



$$\Rightarrow \quad \boxed{W y_i = \frac{1}{n} y_i \cdot y_i^T y_i - \frac{p}{n} y_i}$$

$$y_i^T y_i = \begin{pmatrix} \mp 1 & \mp 1 & \cdots & \mp 1 \end{pmatrix} \begin{pmatrix} \mp 1 \\ \vdots \\ \mp 1 \end{pmatrix} = n$$

$$\Rightarrow \quad \boxed{W y_i = \left(1 - \frac{p}{n}\right) y_i}$$

Result! y_i becomes an EIGENVECTOR of W with an EIGENVALUE $1 - \frac{p}{n}$!

y_i's are orthogonal $\Rightarrow$ $p \leq n$ $\Rightarrow$ $1 - \frac{p}{n} \geq 0$.

$$\Rightarrow \quad \boxed{\text{sign}(W y_i) = \text{sign}\left(\left(1 - \frac{p}{n}\right) y_i\right) = \text{sign}(y_i) = y_i \, !}$$

$\Rightarrow$ All y_i become FIXED POINTS of W!

$\Rightarrow$ Solves problem 1 when patterns are ORTHOGONAL!

NOTE! : Since $\theta = 0$, $o(k+1) = \Gamma(W o(k))$

$\Gamma(W y_i) = y_i \iff \Gamma(W(-y_i)) = \Gamma(-W y_i) = -\Gamma(W y_i) = y$

RESULT: y_i is stored $\iff$ $-y_i$ is stored (False pattern storage)

+ Last problem: Minimum of "computational cost"?

$$\boxed{E = -\frac{1}{2} o^T W o + \theta^T o = -\frac{1}{2} o^T W o}$$

W is symmetric $\Rightarrow$ All eigenvalues are REAL $(\lambda_1, \lambda_2, \ldots, \lambda_n)$

$\lambda_{min} = \min\{\lambda_1, \ldots, \lambda_n\}$, $\lambda_{max} = \max\{\lambda_1, \ldots, \lambda_n\}$

$\Rightarrow \lambda_{min} \|o\|^2 \leq o^T W o \leq \lambda_{max} \|o\|^2$ $\forall o$!

$$\Rightarrow \boxed{-\frac{\lambda_{max}}{2} \|o\|^2 \leq -\frac{1}{2} o^T W o \leq -\frac{\lambda_{min}}{2} \|o\|^2}$$

Eigenvalues of W?

$W y_i = (1 - \frac{p}{n}) y_i$ $i = 1, 2, \ldots, p$; $\Rightarrow$ $\boxed{\lambda_1 = \lambda_2 = \ldots = \lambda_p = 1 - \frac{p}{n}}$

Rest of the eigenvalues?

Let $y_{p+1}, \ldots, y_n$ be ORTHOGONAL COMPLEMENT of $\{y_1, \ldots, y_p\}$.

$$\boxed{y_i^T y_j = 0 \quad i = 1, 2, \ldots, p \quad \forall j = p+1, \ldots, n.}$$

$$W y_j = \left(\frac{1}{n} \sum_{i=1}^{p} y_i y_i^T - \frac{p}{n} I\right) y_j = \frac{1}{n} \sum_{i=1}^{p} y_i y_i^T y_j - \frac{p}{n} y_j = -\frac{p}{n} y_j$$

$\Rightarrow$ $\boxed{W y_j = -\frac{p}{n} y_j}$ $j = p+1, \ldots, n$.

$$\Rightarrow \boxed{\lambda_{p+1} = \ldots = \lambda_n = -\frac{p}{n}}$$

$$\Rightarrow \quad \boxed{\lambda_{max} = 1 - \frac{p}{n} > 0}\, , \quad \boxed{\lambda_{min} = -\frac{p}{n} < 0}$$

$$\Rightarrow \quad -\frac{\left(1-\frac{p}{n}\right)}{2}\|o\|^2 \leq -\frac{1}{2} o^T W o \leq +\frac{p}{2n} \|o\|^2$$

But since $o = \begin{pmatrix} \mp 1 \\ \vdots \\ \mp 1 \end{pmatrix} \Rightarrow \|o\|^2 = (\mp 1 \cdots \mp 1)\begin{pmatrix} \mp 1 \\ \vdots \\ \mp 1 \end{pmatrix} = n$

$$\Rightarrow \quad \boxed{\frac{p-n}{2} \leq -\frac{1}{2} o^T W o \leq \frac{p}{2}}$$

$$E(y_i) = -\frac{1}{2} y_i^T W y_i = -\frac{1}{2} y_i^T \left(1 - \frac{p}{n}\right) y_i = -\frac{1}{2}\left(1 - \frac{p}{n}\right)\|y_i\|^2$$

$$= -\frac{1}{2}(n-p) = \frac{p-n}{2} \quad \leftarrow \text{minimum of } E! \qquad (i = 1, 2, \ldots, p)$$

RESULT: If the patterns are ORTHOGONAL $\Rightarrow$ this design technique satisfies 1-4

Expectation: If the patterns are NEARLY ORTHOGONAL ($y_i^T y_j \approx 0$) $\Rightarrow$ we expect that the method still works

A TRICK: Let $\{y_1, \ldots, y_p\}$ be given with $p < n$.

$\Rightarrow$ Make a transform $y_i \Leftrightarrow \hat{y}_i$ such that $\hat{y}_i$'s are ORTHOGONAL $\Rightarrow$ Use $\hat{y}_i$'s for storage

EXAMPLE:

$$y_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad y_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \qquad y_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$W = \frac{1}{4}\left(y_1 y_1^T + y_2 y_2^T + y_3 y_3^T \right) - \frac{3}{4} I = \frac{1}{4} \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

$$W y_1 = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4} y_1 \quad , \quad W y_2 = \frac{1}{4} y_2 \quad , \quad W y_3 = \frac{1}{4} y_3 .$$

$$E = -\frac{1}{2} o^T W o \quad \Rightarrow \quad \boxed{-\frac{1}{2} = \frac{p-n}{2} \leq E \leq \frac{p}{n} = \frac{3}{2}}$$

Let $y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ be given. (y is "close" to y_1)

$$E(y) = -\frac{1}{2} y^T W y = -\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ -1 \\ 1 \end{pmatrix} = 0$$

synchronous update:

$$v(0) = W y = \frac{1}{4} \begin{pmatrix} 3 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad o(1) = \Gamma(v(0)) = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$v(1) = W o(1) = \frac{1}{4} \begin{pmatrix} -3 \\ 3 \\ 3 \\ -3 \end{pmatrix} \quad \Rightarrow \quad o(2) = \Gamma(v(1)) = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\underset{E=0}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}} \rightarrow \underset{E=0}{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}} \rightarrow \underset{E=0}{\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}}$$

Asynchronous update:

$$\begin{pmatrix}1\\1\\1\\-1\end{pmatrix} \xrightarrow[o_1]{\text{update}} \begin{pmatrix}1\\1\\1\\-1\end{pmatrix} \xrightarrow[o_2]{\text{update}} \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \xrightarrow[o_3]{\text{update}} \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

$$\xrightarrow[o_4]{\text{update}} \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \xrightarrow[o_1]{\text{update}} \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \xrightarrow[o_2]{\text{update}} \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \xrightarrow[o_3]{\text{update}} \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \xrightarrow{o_4} \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

end of cycle

end of cycle.

RESULT: $\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \to -y_3$! (close to y!)

Cost: $E = -\frac{1}{2} y_3^T W y_3 = -\frac{1}{2}$ minimum

Change the order of update:

$$y = \begin{pmatrix}1\\1\\1\\-1\end{pmatrix} \xrightarrow[o_4]{\text{update}} \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \xrightarrow[o_3]{\text{update}} \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \xrightarrow[o_2]{\text{update}} \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \xrightarrow[o_1]{\text{update}} \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

end of cycle.

$$\xrightarrow[o_4]{\text{update}} \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \xrightarrow{o_3} \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \xrightarrow{o_2} \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \xrightarrow{o_1} \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

end of 2 cycle.

RESULT: $\Rightarrow \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = y_1$!

cost: $E = -\frac{1}{2} y_1^T W y_1 = -\frac{1}{2}$: minimum.

* Generalization.

$$\left.\begin{array}{l} v(k) = W o(k) - \theta \\ o(k+1) = \Gamma(v(k)) \end{array}\right\} \Rightarrow \boxed{o(k+1) = \Gamma(W o(k) - \theta)}$$

Given patterns $y_1, \ldots, y_p \in \mathbb{R}^n$

For fixed point property (problem 1)

choose $\theta = 0$.

$y_i = \Gamma(W y_i) \qquad i = 1, 2, \ldots, p.$

$A = [y_1 \; y_2 \cdots y_p] \in \mathbb{R}^{n \times p}$ ($n \times p$ matrix)

$$\Rightarrow \boxed{A = \Gamma(WA)} \Leftrightarrow \boxed{WA = \Gamma^{-1}(A)}$$

call $\boxed{P = \Gamma^{-1}(A)} \Rightarrow \boxed{WA = P}$

properties of P:

i) row space of P is in the row space of A

$$\operatorname{rank}(A) = \operatorname{rank}\begin{pmatrix} A \\ P \end{pmatrix}$$

ii) $\operatorname{sgn}(a_{ij}) = \operatorname{sgn}(p_{ij})$

conversely, for any P satisfying i) and ii), we can find a suitable A.

Typical choices for P:

i) $P = DA$, $D = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$ $\alpha_i > 0$

ii) $\alpha_1 = \alpha_2 = \dots = \alpha_n = \tau_1 \Rightarrow P = \tau_1 A$, $\tau_1 > 0$.

In general, let us use Singular Value Decomposition (SVD)

$\text{rank}(A) = r < p$, $A \in \mathbb{R}^{n \times p}$

$$A = U \Sigma V^T$$

U: orthogonal matrix $(UU^T = U^T U = I) \in \mathbb{R}^{n \times n}$

$$U = [\underbrace{U_1}_{r} \ \underbrace{U_2}_{n-r}] \}n$$

$U_1: n \times r$
$U_2 = n \times (n-r)$

V: orthogonal matrix $(VV^T = V^T V = I) \in \mathbb{R}^{p \times p}$

$$V = [\underbrace{V_1}_{r} \ \underbrace{V_2}_{p-r}] \}p$$

$V_1 = p \times r$
$V_2 = p \times (p-r)$

$$\Sigma = \begin{array}{c} \begin{pmatrix} \Sigma_1 & 0 \\ \underbrace{0}_{r} & \underbrace{0}_{n-r} \end{pmatrix} \begin{array}{l} \}r \\ \}p-r \end{array} \end{array}$$

$\Sigma: n \times p$
$\Sigma_1: r \times r$

$$\Sigma_1 = \begin{pmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_r \end{pmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$

σ_i: singular values of A.

$$AA^T = U\Sigma V^T V \Sigma^T U^T = U \Sigma \Sigma^T U^T$$

$$= U \begin{pmatrix} \Sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix} U^T$$

$$\Sigma_1^2 = \begin{pmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_r^2 \end{pmatrix}$$

$$\Rightarrow \boxed{\sigma_i = \sqrt{\text{nonzero eigenvalue of } AA^T} \quad i = 1, 2, \ldots r}$$

$$A = U\Sigma V^T = (U_1 \; U_2) \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix}$$

$$\boxed{A = U_1 \Sigma_1 V_1^T}$$

$$\Rightarrow \quad WA = P$$

$$\Rightarrow \quad WA = W U_1 \Sigma_1 V_1^T = P$$

$$\Rightarrow \quad W U_1 \Sigma_1 V_1^T V_1 = P V_1 \qquad (V_1^T V_1 = I)$$

$$\Rightarrow \quad W U_1 \Sigma_1 = P V_1$$

$$\Rightarrow \quad \boxed{W U_1 = P V_1 \Sigma_1^{-1}}$$

$$\boxed{W U_2 = X} \qquad \underline{X: n \times (n-r) \text{ and ARBITRARY!}}$$

$$\Rightarrow \quad W(U_1 \; U_2) = (P V_1 \Sigma_1^{-1} \;\vdots\; X)$$

$$W = (P\, V_1 \Sigma_1^{-1} : X) \begin{pmatrix} U_1^T \\ U_2^T \end{pmatrix}$$

$$\boxed{W = P V_1 \Sigma_1^{-1} U_1^T + X\, U_2^T}$$

<u>CONCLUSION:</u> For any P satisfying i), ii) and for any arbitrary X, this W solves the fixed point problem! (solves problem 1).

<u>IDEA:</u> Manipulate P and X to satisfy problems 2-4.

A special choice:

$$P = \tau_1 A, \qquad X = -\tau_2 U_2 \qquad \tau_1 > 0,\ \tau_2 > 0$$

$$\Rightarrow \quad W = \tau_1 A \cdot V_1 \Sigma_1^{-1} U_1^T - \tau_2 U_2 U_2^T$$

$$= \tau_1\, U_1 \Sigma_1 V_1^T\, V_1 \Sigma_1^{-1} U_1^T - \tau_2 U_2 U_2^T$$

$$= \tau_1\, U_1 \Sigma_1 \Sigma_1^{-1} U_1^T - \tau_2 U_2 U_2^T$$

$$\boxed{W = \tau_1 U_1 U_1^T - \tau_2 U_2 U_2^T}$$

3) <u>$W = W^T$!</u> by choosing τ_1 and τ_2 appropriately, we may have $w_{ii} = 0$! (problem 4).

Minimum:

$*)$ $U = [U_1 \vdots U_2]$ $\qquad U^T U = I$

$$\Rightarrow \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} [U_1 \; U_2] = \begin{bmatrix} U_1^T U_1 & U_1^T U_2 \\ U_2^T U_1 & U_2^T U_2 \end{bmatrix}$$

$\Rightarrow$ $U_1^T U_1 = I$, $\quad U_2^T U_2 = I$, $\quad U_1^T U_2 = 0 \quad U_2^T U_1 = 0.$

$*)$ $UU^T = I \Rightarrow (U_1 \; U_2) \begin{pmatrix} U_1^T \\ U_2^T \end{pmatrix} = U_1 U_1^T + U_2 U_2^T = I$

$*)$ $A = U_1 \Sigma_1 V_1^T$ & $A = [y_1 \cdots y_p]$

$\Rightarrow$ y_i are in the column space of U_1, and hence are orthogonal to U_2.

$*$ $y_i^T (U_1 U_1^T + U_2 U_2^T) y_i = y_i^T y_i$

$\Rightarrow$ $y_i^T U_1 U_1^T y_i = y_i^T y_i = \|y_i\|^2 = n$ (binary case!)

$*$ $$\boxed{W = \tau_1 U_1 U_1^T - \tau_2 U_2 U_2^T}$$

$\Rightarrow$ $\lambda_1 = \lambda_2 = \cdots = \lambda_r = \tau_1$ (r eigenvalues are τ_1)

$\lambda_{r+1} = \cdots = \lambda_n = -\tau_2$ (remaining ones are $-\tau_2$)

$\Rightarrow$ $\lambda_{min}(W) \|y\|^2 \leq y^T W y \leq \lambda_{max}(W) \|y\|^2$

$\Rightarrow$ $-\frac{\lambda_{max}(W)}{2} \|y\|^2 \leq -\frac{1}{2} y^T W y \leq -\frac{\lambda_{min}(W)}{2} \|y\|^2$

$$\left.\begin{array}{l} \lambda_{max}(W) = \tau_1 \\ \lambda_{min}(W) = -\tau_2 \\ \|y\|^2 = n \end{array}\right\} \Rightarrow \boxed{-\frac{\tau_1 n}{2} \leq -\frac{1}{2} y^T W y \leq \frac{\tau_2 n}{2}}$$

$$E(y_i) = -\frac{1}{2} y_i^T W y_i = -\frac{1}{2} y_i^T \left(\tau_1 U_1 U_1^T - \tau_2 U_2 U_2^T \right) y_i$$

$$= -\frac{\tau_1}{2} y_i^T U_1 U_1^T y_i + \frac{\tau_2}{2} y_i^T U_2 U_2^T y_i$$

$$= -\frac{\tau_1}{2} \|y_i\|^2 = -\frac{\tau_1 n}{2}.$$

$\boxed{\text{Conclusion: } y_i\text{'s become minima of cost}}$

solves problem 4!

Relation with outer product Rule.

$$\boxed{W = \frac{1}{n} \sum_{i=1}^{p} y_i y_i^T - \frac{p}{n} I}$$

Assume that y_i's are ORTHOGONAL.

$$A = [y_1\ y_2 \cdots y_p] \qquad \|y_i\| = \sqrt{(\pm 1)^2 + \cdots (\pm 1)^2} = \sqrt{n}$$

$$A = \underbrace{\left[\frac{y_1}{\sqrt{n}}\ \frac{y_2}{\sqrt{n}} \cdots \frac{y_p}{\sqrt{n}} \right]}_{U_1} \underbrace{\begin{bmatrix} \sqrt{n} & & 0 \\ & \ddots & \\ 0 & & \sqrt{n} \end{bmatrix}}_{\Sigma_1} \underbrace{I}_{V_1^T}$$

$$\Rightarrow \boxed{U_1 = \frac{1}{\sqrt{n}} A} \qquad \boxed{U_1 U_1^T + U_2 U_2^T = I}$$

$$W = \frac{1}{n} \sum_{i=1}^{p} y_i y_i^T - \frac{p}{n} I = \frac{1}{n} A A^T - \frac{p}{n} I$$

$$= \frac{1}{\sqrt{n}} A \cdot \frac{1}{\sqrt{n}} A^T - \frac{p}{n} I$$

$$= u_1 u_1^T - \frac{p}{n} (u_1 u_1^T + u_2 u_2^T)$$

$$= (1 - \frac{p}{n}) u_1 u_1^T - \frac{p}{n} u_2 u_2^T$$

$$= \tau_1 u_1 u_1^T - \tau_2 u_2 u_2^T$$

Same as outer product rule with $\boxed{\tau_1 = 1 - \frac{p}{n} \, , \tau_2 = -\frac{p}{n}}$

What if the patterns are not orthogonal?

Can we still use outer-product rule?

<u>A Trick</u>: Dummy augmentation:

$$y_i = \begin{pmatrix} y_{i1} \\ \vdots \\ y_{in} \end{pmatrix} \qquad y_j = \begin{pmatrix} y_{j1} \\ \vdots \\ y_{jn} \end{pmatrix}$$

$\Rightarrow$ add dummy entries.

$$\hat{y}_i = \begin{pmatrix} y_{i1} \\ y_{in} \\ \left.\begin{matrix} y_{i\,n+1} \\ y_{i\,m} \end{matrix}\right\} \end{pmatrix} \qquad \hat{y}_j = \begin{pmatrix} y_{j1} \\ y_{jn} \\ \left.\begin{matrix} y_{j\,n+1} \\ y_{jm} \end{matrix}\right\} \end{pmatrix}$$

Dummy entries

such that $\hat{y}_i \perp \hat{y}_j$.

What happens if the patterns are not orthogonal?

$$\boxed{W = \frac{1}{n} AA^T - \frac{P}{n} I}$$

$$WA = \frac{1}{n} A A^T A - \frac{P}{n} A = P$$

$$\Rightarrow \quad P = \left(\frac{1}{n} AA^T - \frac{P}{n} I\right) A$$

$\Rightarrow$ row space of P is in row space of A!

$$\text{sign}(P_{ij}) = \text{sign}(a_{ij}) \ ?$$

$$A = [y_1 \ y_2 \cdots y_P]$$

$$A^T A = \begin{bmatrix} y_1^T y_1 & y_1^T y_2 & \cdots & y_1^T y_P \\ \vdots & & & \\ y_P^T y_1 & \cdots & & y_P^T y_P \end{bmatrix}$$

$$y_i^T y_i = \|y_i\|^2 = n. \qquad y_i^T y_j = ?$$

$$(y_{i1} \cdots y_{in}) \begin{pmatrix} y_{j1} \\ \vdots \\ y_{jn} \end{pmatrix} = y_{i1} y_{j1} + \cdots + y_{in} y_{jn}$$

Assume that h_{ij} bits are different, and $n - h_{ij}$ bits are the same.

$$\boxed{y_i^T y_j = n - h_{ij} - h_{ij} = n - 2 h_{ij}}$$

h_{ij} : Hamming distance between y_i and y_j

$$P = \frac{1}{n} AA^TA - \frac{p}{n} A = \frac{1}{n} A \left(A^TA - I \right)$$

$$\begin{bmatrix} p_{11} \cdots p_{1n} \\ p_{ij} \\ \cdots \end{bmatrix} = \frac{1}{n} \begin{bmatrix} a_{11} \cdots a_{1n} \\ a_{ij} \\ \cdots \end{bmatrix} \begin{bmatrix} n-p & n-2h_{12} & n-2h_{1p} \\ & n-p & \cdots \\ \cdots & & n-p \end{bmatrix}$$

$\Rightarrow$ sign (a_{ij}) = sign (p_{ij}) is guaranteed if we have:

$$\boxed{\sum_{\substack{l=1 \\ l \neq j}}^{p} |n - 2h_{lj}| < n-p} \Leftrightarrow$$

the matrix $AA^T - pI$ is DIAGONALLY DOMINANT!

$$p_{ij} = a_{i1}(n - 2h_{1j}) + a_{i2}(n - 2h_{2j}) + \cdots + a_{ij}(n-p) + \cdots$$

$$a_{ij} = +1 \rightarrow p_{ij} = (n-p) + \sum_{\substack{k=1 \\ k \neq j}}^{p} a_{ik}(n - 2h_{kj}) \geq (n-p) - \sum_{\substack{k=1 \\ k \neq j}}^{p} |n - 2h_{kj}| > 0$$

$$a_{ij} = -1 \quad p_{ij} = -(n-p) + \sum_{\substack{k=1 \\ k \neq j}}^{p} a_{ik}(n - 2h_{kj}) \leq -(n-p) + \sum_{\substack{k=1 \\ k \neq j}}^{p} |n - 2h_{kj}| < 0$$

hence in both cases sign (a_{ij}) = sign (p_{ij})