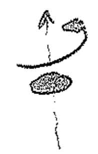
SPIN GLASS THEORY

— Spin is related to rotation of any particle. Usually rotation speed is fixed, and we are concerned with ONLY the orientation.



→ Right rotation ⇒ up state ↑

— left rotation ⇒ down state ↓

→ Hence could be represented by a BINARY VARIABLE σ_x

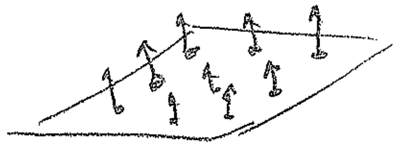
→ $\sigma_x = +1 \Leftrightarrow$ Right/up rotation of particle x

$\sigma_x = -1 \Leftrightarrow$ left/down rotation of particle x.

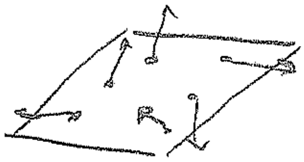
— Same "spin" rotation could be applied to any sub-atomic particle as well

— Closely related to "magnetic" properties.

— Good magnetization ⇒ when all spins have the same direction ⇒ ferromagnetism



— Poor magnetization ⇒ when spins have different directions ⇒ antiferromagnetism



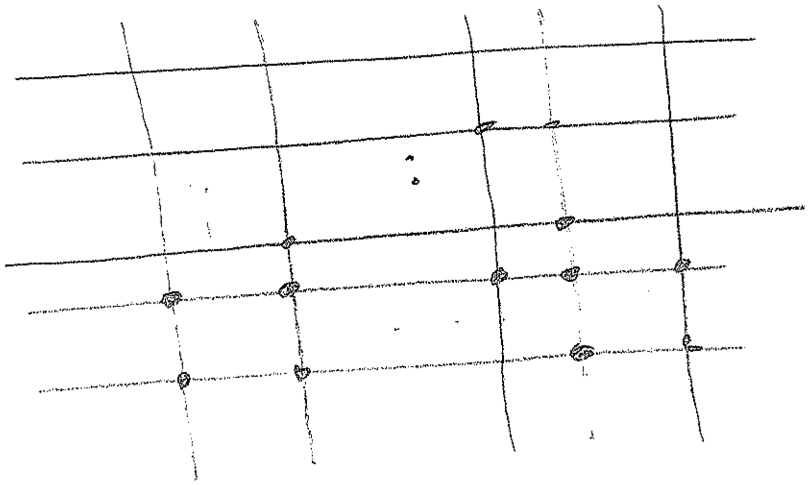
— Why "glass"? Good magnetization ⇔ quartz (crystalline) structure

poor magnetization ⇔ ordinary (amorphous) glass

— Similar to different "glass" structures, different "spin" or "magnetization" structures are possible.

— How to deal with "spin" or "magnetization"?

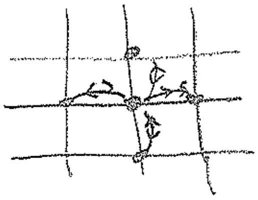
— Assume that there are N such particles, each having a "spin state" $\sigma_i \in \{-1, 1\}$, $i = 1, 2, \dots, N$, in a given structure. For example, consider 2 dimensional lattice.



particles "interact" with each other according to a "strength" J_{ij}. If there is an external magnetic field $\vec{H}$ ⇒ Energy of the configuration is given as

$$\boxed{J(\sigma) = -\sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}\,\sigma_i \sigma_j - \sum_{i=1}^{N} h_i \sigma_i}$$

— <u>Assumption</u>: Each particle only interacts with its nearest neighbor (near field theory)



— <u>Assumption</u>: $J_{ij} = J_{ji}$ (symmetry!)

— <u>Assumption</u>: $J_{ii} = 0$ (no self-interaction)

— When $J_{ij} > 0 \Rightarrow$ ferromagnetic interaction between particles i and j

— When $J_{ij} < 0 \Rightarrow$ antiferromagnetic interaction between particles i and j.

— <u>BASIC PHYSICAL RULE</u>: Given J_{ij} and h_i, the only possible spin configurations σ are the LOCAL MINIMA of $J(\sigma)$ (called Hamiltonian)

<u>PROBLEM</u>: Given interactions J_{ij}, h_i, find the possible spin configurations, which are

$$\min J(\sigma)$$

σ^*: Global minimum $\Leftrightarrow$ GROUND STATE.

PROBLEM: How to find σ^*? Very Hard problem (NP-complete)

Related problems:

- Given two possible spin configurations σ^A, σ^B, can there be a phase transition?
  ($\sigma^A \to \sigma^B$ or vice versa)

- Given a DESIRED spin configuration σ^D, how can we find appropriate connection strengths J_{ij} such that σ^D becomes a local (global?) minimum of $J(\sigma)$?
  (i.e. $J(\sigma^D) \le J(\sigma^S)$ for any σ^S configuration)

$\Rightarrow$ DESIGN PROBLEM.

- What if J_{ij}'s are random variables?
  $\Rightarrow$ (mean field theory)

- What if J_{ij}'s depend on temperature and temperature is changing?
  ($\Rightarrow$ Boltzmann machine)
  $\Rightarrow$ glass analogy)

- Can we detect stable and unstable spin configurations?

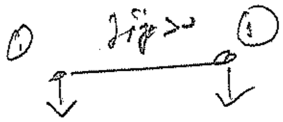
- Can we model transitions between spin states?
  (Time-scale problem ..)

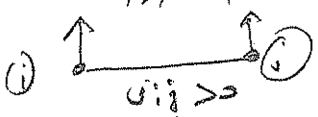
— General considerations / observations

If $J_{ij} > 0$ (ferromagnetic coupling) then $\sigma_i \sigma_j > 0$ should hold to minimize $-\sum\sum J_{ij} \sigma_i \sigma_j$

$\Leftrightarrow$ maximize $\sum\sum J_{ij} \sigma_i \sigma_j$

$\Rightarrow$ ferromagnetic materials tend to orient the spins in the same direction.

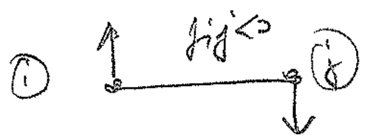


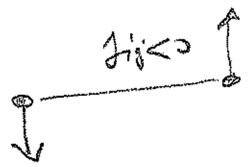


— If $J_{ij} < 0$ (antiferromagnetic coupling) then $\sigma_i \sigma_j < 0$ should hold to minimize $-\sum\sum J_{ij} \sigma_i \sigma_j$

($\Leftrightarrow$ maximize $\sum\sum J_{ij} \sigma_i \sigma_j$)

$\Rightarrow$ In antiferromagnetic materials, spins are not oriented in the same direction.



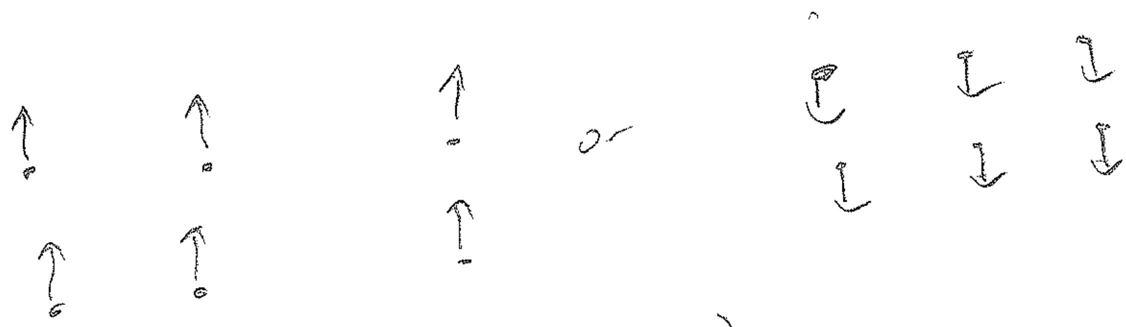


(h=0 case)

⇒ If all $J_{ij} > 0$ ⇒ All $\sigma_i = +1$ gives GROUND STATE

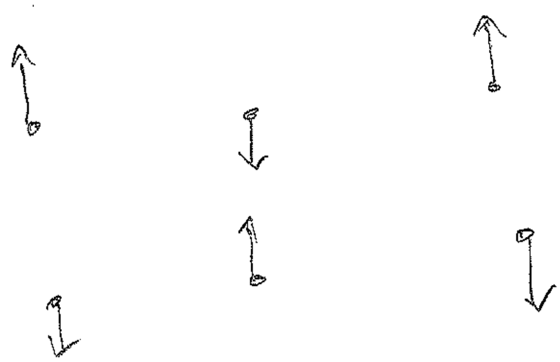
All $\sigma_i = -1$ " " "

Eg



⇒ PERFECT MAGNET (ferromagnetism)

If all $J_{ij} < 0$ ⇒ GROUND STATE is as follows

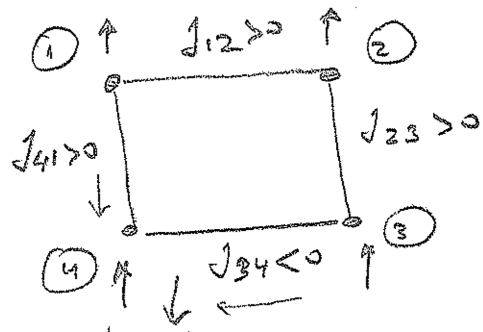


or its mirror image!

What if some $J_{ij} > 0$ and some others $J_{ij} < 0$

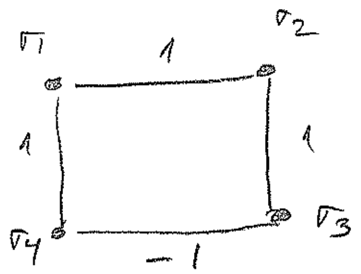
(impurities — defects?)

This may yield to "conflicts", or so-called "FRUSTRATIONS" if ferromagnetic and antiferromagnetic connections occur in the same chain



Both states are possible. Which one do we observe? $\Rightarrow$ FRUSTRATION (Two probable spin configurations) Two local minima)

Example:



$$-J(\sigma) = \sigma_1\sigma_2 + \sigma_2\sigma_3 - \sigma_3\sigma_4 + \sigma_4\sigma_1$$

PROBLEM: Find $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \{-1, +1\}$ such that $-J(\sigma)$ is maximum ($\Leftrightarrow J(\sigma)$ is minimum)

Intuitively $\sigma_1 = \sigma_2 = \sigma_3 = 1$. Keep σ_4 as unknown

$$\Rightarrow \boxed{-J(\sigma) = 1 + 1 - \sigma_4 + \sigma_4 = 2}$$

INDEPENDENT OF σ_4!

i.e. $\sigma_4 = +1$, $\sigma_4 = -1$ $\Rightarrow$ SAME energy level!

A spin glass is a structure (magnet / glass) with FRUSTRATED INTERACTIONS ($J_{ij} > 0$ and $J_{ij} < 0$) and random DISORDERS (J_{ij}'s may be random)

— RELATION WITH TEMPERATURE and STATISTICAL PHYSICS

→ Assume that system is in a spin configuration σ^s at a temperature T. The Probability of observing σ^s at T is given by

$$\boxed{P(\sigma = \sigma^s) = C \cdot e^{-\frac{J(\sigma)}{kT}}}$$

k : Boltzmann constant, $J(\sigma)$: spin Hamiltonian

C : normalization constant

$$\Rightarrow \sum_s P(\sigma = \sigma^s) = C$$

— ORDER PARAMETER ⟺ magnetization :

$$\boxed{m = \frac{1}{N} \sum_{i=1}^{N} \sigma_i}$$

— If all $\sigma_i = 1$ ⇒ $m = 1$ (perfect magnet)
— If all $\sigma_i = -1$ ⇒ $m = -1$ (" ")
— If #+1 ≈ #−1 → $m = 0$ (up-down states cancel each other)
POOR MAGNET.

IDEA: At low temperatures, low energy states are favorable
at high temperatures, all energy levels are possible.

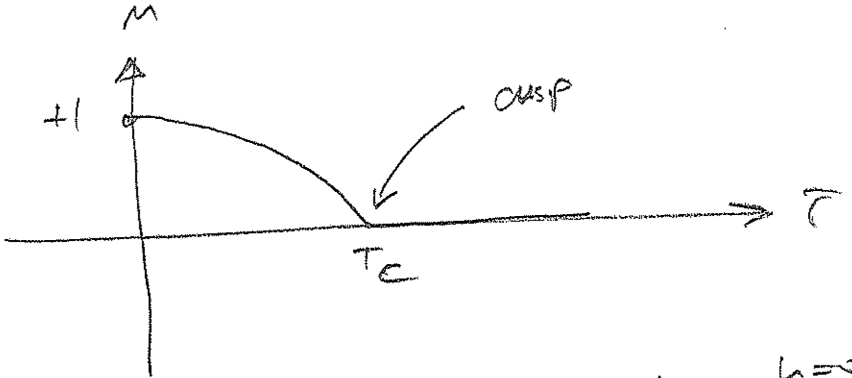
Assume $J_{ij} > 0$ (ferromagnet)

– At low temperature → Boltzmann distribution favors low energy levels ⇒ all spin up / down case

⇒ $m = \mp 1$ ⇒ PERFECT MAGNET (at $T=0$)

– At high temperature ⇒ All σ's configurations are equally likely ⇒ $m \to 0$.

BUT! The change of m as temperature increases is NOT linear! It has a CUSP Bifurcation.



This is a typical diagram when $h = 0$ (no external field)

T_c : phase transition temperature.

when $T < T_c$: ferromagnetic behavior
when $T > T_c$: paramagnetic behavior

– Various interesting phase transitions occur for $T > T_c$ and $T < T_c$ case

– Relation with $\underline{h}$ is also important

– Still under investigation.

– What if impurities exist ($J_{ij} < 0$ for some bondings)?

⇒ SPIN GLASS STATE.

Typically

FERROMAGNETS:

$J_{ij} = J_0 + r(T)$ $\quad J_0 > 0$

$r(T)$: random noise, depending on temperature fluctuations

and as $T \to 0$, $r(T) \to 0$

$\Rightarrow$ at $T=0$ $\quad J_{ij} = J_0 > 0$ $\Rightarrow$ ground state: perfect magnet.

PARAMAGNETS

Same with $J_0 < 0$

as $T \to 0$, $\quad J_{ij} = J_0 < 0$ $\to$ no magnetization.

SPIN GLASS CASE:

$J_{ij} = J_{ij0} + r(T)$ $\quad J_{ij0} > 0$ or $J_{ij0} < 0$

although as $T \to 0$, $r(T) \to 0$, sign (J_{ij}) is RANDOM.

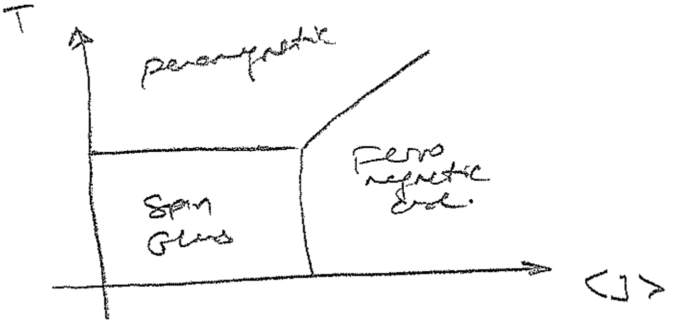
so spins "freezes" as $T \to 0$ but in a random way

$\Rightarrow$ as $T \to 0$, unusually large type of configurations are possible, having different stability properties

$\Rightarrow$ as $T \uparrow$ and $T \downarrow$, due to randomness, the behaviour will not be the same $\Rightarrow$ Hysteresis behaviour

What if some $J_{ij} < 0$, some $J_{ij} > 0$ (disordered)

$\Rightarrow$ Behavior depends on $\langle J \rangle$ (average of J_{ij}) and T



In Spin-Glass case, large number of possible σ^s configurations with equal $J(\sigma^s)$, and similar stability properties can be observed ($\Rightarrow$ large # of local minima / local maxima). phase transitions can also be observed.

RELATION WITH NEURAL NETWORKS (HOPFIELD)

(1) — $$o(k+1) = \Gamma(W o(k) - \theta)$$

- In Hopfield case $W = W^T$
- If $w_{ii} = 0$ $\Rightarrow$ asynchronous update is useful

— $$\min \; -\frac{1}{2} o^T W o + \theta^T o$$

$$J(o) = -\sum\sum w_{ij} o_i o_j + \sum \theta_i o_i$$

tries to achieve an output configuration o where $J(o)$ is locally minimum.

— Similar to Spin-Glass theory.

— Could be used to determine the local minima of spin glasses.

(2) — $J_{ij} > 0$ ferromagnetic behaviour $\Leftrightarrow$ exciting synapse

$J_{ij} < 0$ antiferromagnetic behaviour $\Leftrightarrow$ inhibitory synapse.

— Spin glass disorder can be analyzed similarly by Neural network modeling

(3) Local minima $\Leftrightarrow$ pattern storage / retrieval
Could be applied to analyze spin glass states.

(4) Transition behavior can be analyzed. (for spin glasses)

(5) If we want some spin configurations $\sigma^1, \ldots, \sigma^p$ to be possible $\Rightarrow$ a candidate interaction

$$J = \frac{1}{N} \sum_{i=1}^{p} \sigma^i \sigma^{iT} \Rightarrow \text{Hebbian learning Rule}$$

(Conversely if there is a connection $J \Rightarrow$ possible spin configurations can be analyzed)

(6) Analogy helps us to carry the developments from one field to another

(7) Could help us to build statistical theory
$\Rightarrow$ Boltzmann machines

$\Longleftrightarrow$