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EEE 443/543 Neural Networks
Midterm, Fall 2012-13

No credits will be given for unjustified answers.

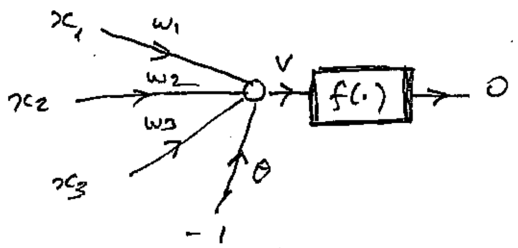
Prob. 1 : (30 pt.s) Let $x_1,\ x_2,\ x_3$ be 3 logic variables such that $x_i = \{0,\ 1\}$, $i = 1, 2, 3$. We want to realize the following logic function o :

$$o = \begin{cases} 0 & x_1 = x_2 = x_3 \\ 1 & \text{otherwise} \end{cases}$$

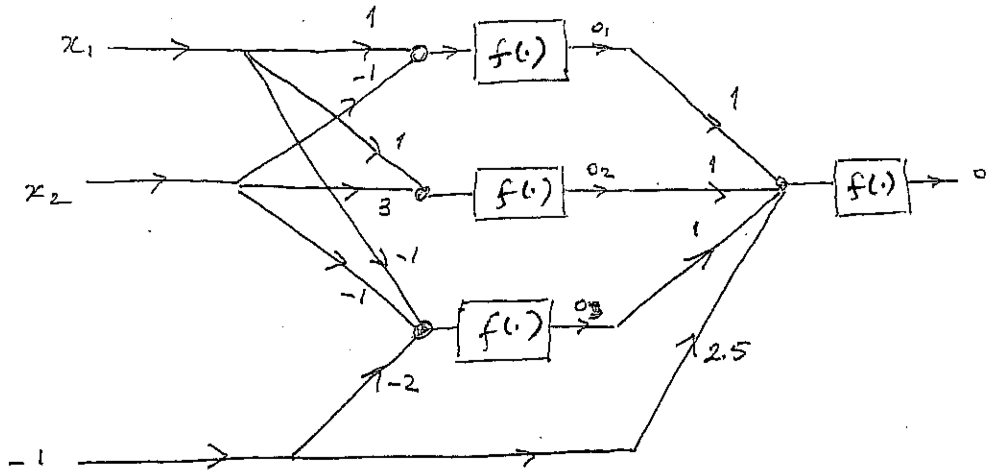
Now consider the standard neuron model as given below, where the nonlinear function $f(v)$ is a hard limiter (i.e. $f(v) = 1$ when $v \geq 0$, and $f(v) = 0$ when $v < 0$).

i : Show analytically that by using a single neuron as given below, the logic function o defined above cannot be realized.

ii : We know that any logic function can be realized by using multi layer structures. By using the type of neurons given below, design a multi layer neural network which realizes the logic function o defined above.



Prob. 2 : (20 pt.s) Consider the following network. Note that as usual, $v = w_1x_1 + w_2x_2 - \theta$, and the nonlinear function $f(v)$ is the hard limiter function, i.e. $f(v) = 1$ when $v \geq 0$ and $f(v) = 0$ when $v < 0$. Find the region at which $o = 1$, and indicate this region geometrically in $x_1 - x_2$ plane. The weights are indicated on the network.



Prob. 3 : (30 pt.s) Let x_1, x_2 denote two scalar inputs, and set $z = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathrm{R}^2$. Consider the single neuron given below; as usual the scalar output o is given by $o = sgn(w_1 x_1 + w_2 x_2 - \theta)$; here w_1, w_2 are scalar weights, θ is is the scalar threshold value and and $f = sgn$ is the signum function given as $f(v) = 1$ when $v \geq 0$ and $f(v) = -1$ when $v < 0$. Let us define the (unextended) input pattern vectors as :

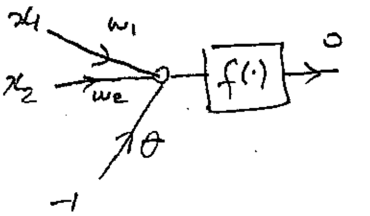
$$z_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ z_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix},\ z_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ z_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix},\ z_5 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ z_6 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

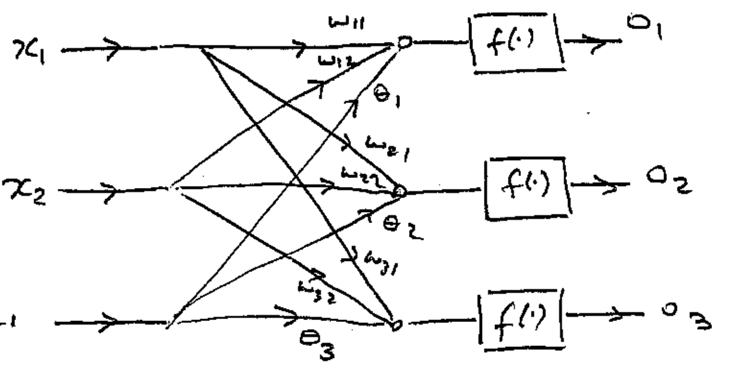
Now consider the 3 classes given as $C_1 = \{z_1, z_2\}$, $C_2 = \{z_3, z_4\}$, $C_3 = \{z_5, z_6\}$. Our aim is to design a single layer network with 3 outputs o_1, o_2, o_3 such that $o_i = 1 \Longleftrightarrow z \in C_i$, and $o_i = -1$ otherwise, $i = 1, 2, 3$. For this purpose perceptron training algorithm for multi category case will be used.

i : Consider the structure given below, use learning coefficient $c = 1$, and choose the initial weights as $w_{i1} = w_{i2} = 1$, $\theta_i = 0$, $i = 1, 2, 3$. Now use the perceptron training algorithm for multi category case only for one epoch. Find the final weights at the end of one epoch.

ii : Use the last updated weights found above, draw the lines $w_{i1}x_1 + w_{i2}x_2 - \theta_i = 0$ on $x_1 - x_2$ plane, $i = 1, 2, 3$, and decide **geometrically** on whether the classification problem given above is solved or not.

iii : Now consider the structure given below with the last updated weights, apply each pattern from the training set and calculate the corresponding outputs and decide whether the classification problem given above is solved or not.





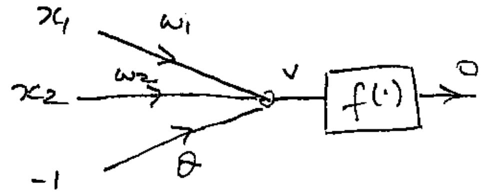
Prob. 4 : (20 pt.s) Consider the following neuron model. Here $v = w_1 x_1 + w_2 x_2 - \theta$, $o = f(v)$, and f is given in the figure shown below. Let the training set patterns (unextended) be given as

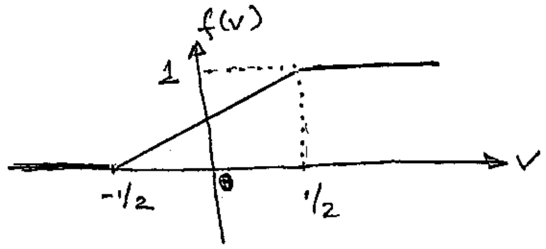
$$z_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The desired outputs are $d = 1$ for z_1 and $d = 0$ for z_2. The cost function to be minimized is the usual quadratic function : $E = 0.5(d - o)^2$.

i : Let the initial weights be given as $w_1(1) = 0$, $w_2(1) = -1.5$, $\theta(1) = 0.25$, and the learning coefficient η as $\eta = 1$. Use the back propagation algorithm (i.e. the gradient descent method) for the given cost function and the neuron model, update the weights (including threshold) for one epoch. Determine the last updated weights (including threshold) at the end of one epoch.

ii : Now consider the weight vector found above. Find the average error over the training set.





EEE 443 MT1 FALL 2012-13 SOLUTIONS.

PROB. 1 :

i) (10/10)

$$O = \begin{cases} 0 & x_1 = x_2 = x_3 \\ 1 & \text{otherwise} \end{cases} \qquad o = f(v) \quad v = w_1x_1 + w_2x_2 + w_3x_3 - \theta$$

(02)

x_1	x_2	x_3	O	
0	0	0	0	$\Rightarrow -\theta < 0 \Rightarrow \theta > 0$
0	0	1	1	$\to w_3 - \theta \geq 0$
0	1	0	1	$\to w_2 - \theta \geq 0$
0	1	1	1	
1	0	0	1	$\to w_1 - \theta \geq 0$
1	0	1	1	
1	1	0	1	
1	1	1	0	$\to w_1 + w_2 + w_3 - \theta < 0$

$w_1 + w_2 + w_3 - 3\theta > 0$ (02)

$w_1 + w_2 + w_3 - \theta \geq 2\theta > 0$ (03)

$w_1 + w_2 + w_3 - \theta < 0$

CONTRADICTION (03)

$\Rightarrow$ Cannot be realized with single neuron.

ii) (20/20)

$O = 1$ when

$x_1 = 0, x_2 = 0, x_3 = 1 \Rightarrow \bar{x}_1 \wedge \bar{x}_2 \wedge x_3 = 1$ OR

$x_1 = 0 \quad x_2 = 1 \quad x_3 = 0 \Rightarrow \bar{x}_1 \wedge x_2 \wedge \bar{x}_3 = 1$ OR

$x_1 = 0 \quad x_2 = 1 \quad x_3 = 1 \Rightarrow \bar{x}_1 \wedge x_2 \wedge x_3 = 1$ OR

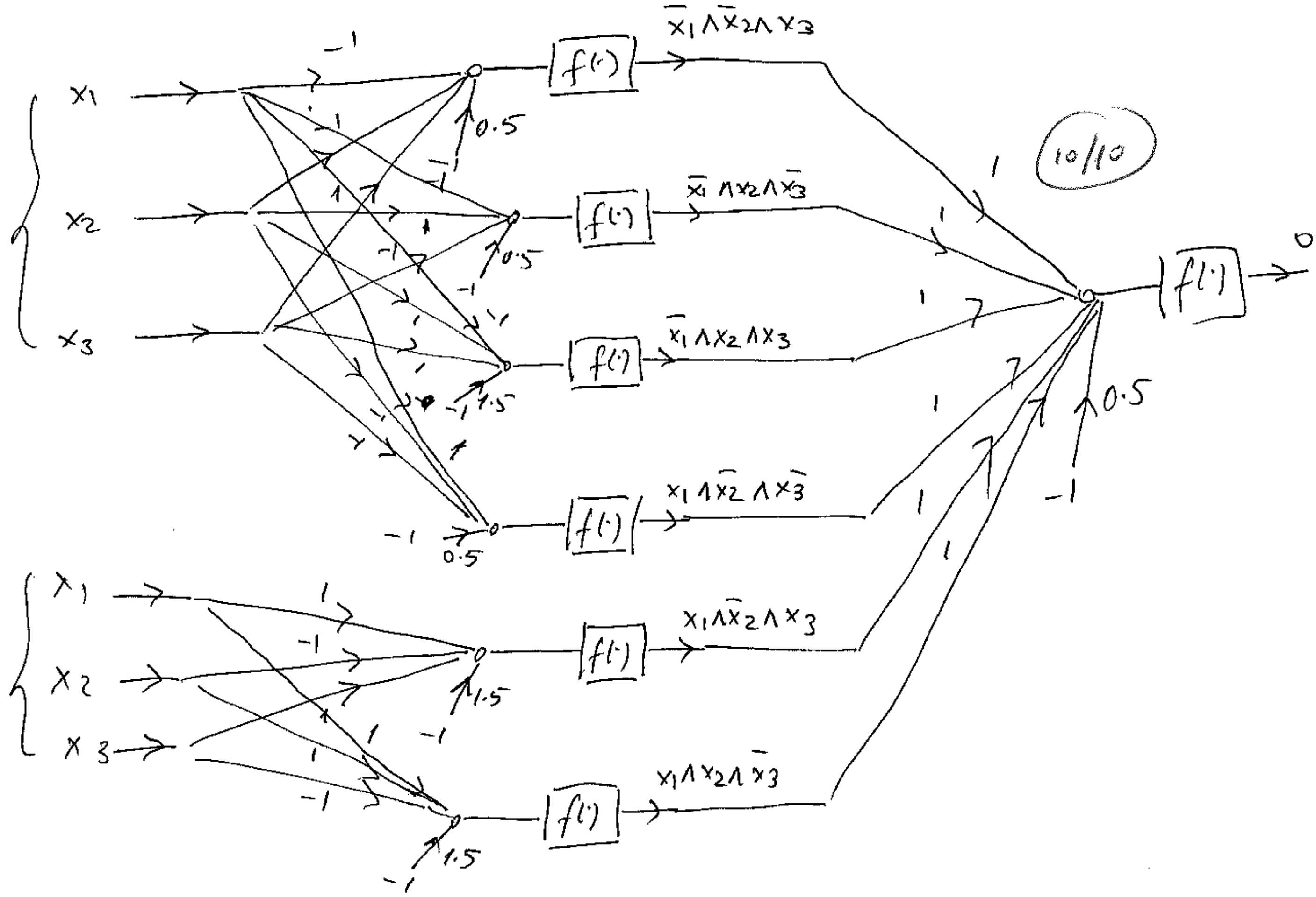
$x_1 = 0 \quad x_2 = 1 \quad x_3 = 0 \Rightarrow x_1 \wedge \bar{x}_2 \wedge \bar{x}_3$ OR

$x_1 = 1 \quad x_2 = 0 \quad x_3 = 0 \Rightarrow x_1 \wedge \bar{x}_2 \wedge x_3$ OR

$x_1 = 1 \quad x_2 = 0 \quad x_3 = 1 \Rightarrow x_1 \wedge x_2 \wedge \bar{x}_3$

$x_1 = 1 \quad x_2 = 1 \quad x_3 = 0$

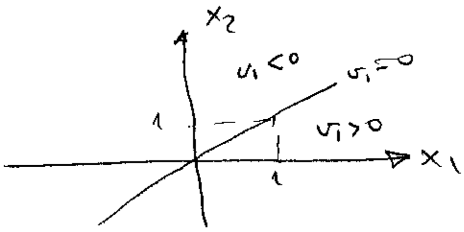
(10/10)



NOTE THAT SUCH A REALIZATION IS NOT UNIQUE.

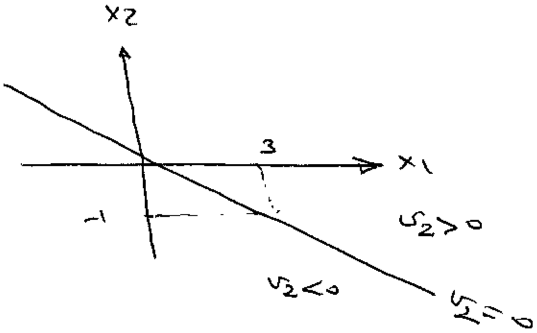
PROB. 2:

$v_1 = x_1 - x_2$



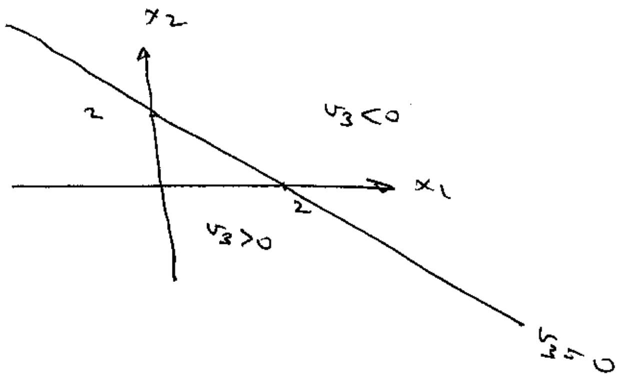
05

$v_2 = x_1 + 3x_2$



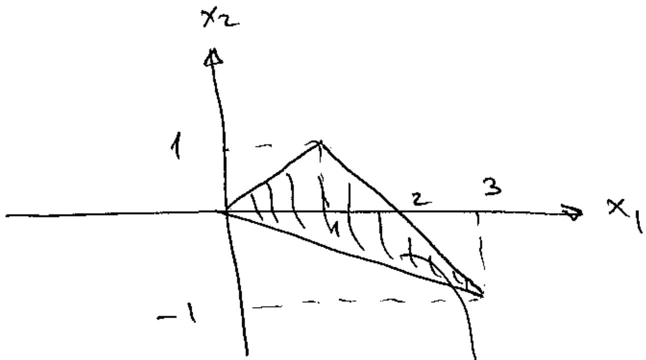
05

$v_3 = -x_1 - x_2 + 2$



05

$o = o_1 \wedge o_2 \wedge o_3 \Rightarrow \begin{array}{l} v_1 \geq 0 \\ v_2 \geq 0 \\ v_3 \geq 0 \end{array}$



05

$o = 1$ HERE

$o = 0$ OUTSIDE

PROB. 3) $w_1(1) = w_2(1) = w_3(1) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

i) $\frac{14}{14}$

$z_1^e = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rightarrow \left.\begin{matrix} v_1 = 1 \Rightarrow o_1 = 1 \\ v_2 = 1 \Rightarrow o_2 = 1 \\ v_3 = 1 \Rightarrow o_3 = 1 \end{matrix}\right\}$ incorrect $w_2(2) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ 02

$w_3(2) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$z_2^e = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \rightarrow \left.\begin{matrix} v_1 = 2 \rightarrow o_1 = 1 \\ v_2 = -1 \rightarrow o_2 = -1 \\ v_3 = -1 \rightarrow o_3 = -1 \end{matrix}\right\}$ correct $\Rightarrow$ No update 02

$z_3^e = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \rightarrow \begin{matrix} v_1 = 1 \rightarrow o_1 = 1 \rightarrow \text{incorrect} \\ v_2 = 0 \rightarrow o_2 = 1 \\ v_3 = 0 \rightarrow o_3 = 1 \rightarrow \text{incorrect} \end{matrix}$ $w_1(3) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ 02

$w_3(3) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

$z_4^e = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \rightarrow \left.\begin{matrix} v_1 = -1 \rightarrow o_1 = -1 \\ v_2 = 1 \rightarrow o_2 = 1 \\ v_3 = -2 \rightarrow o_3 = -1 \end{matrix}\right\}$ correct $\rightarrow$ NO UPDATE 02

$z_5^e = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \rightarrow \begin{matrix} v_1 = -2 \rightarrow o_1 = -1 \\ v_2 = -1 \rightarrow o_2 = -1 \\ v_3 = -2 \rightarrow o_3 = -1 \end{matrix} \Rightarrow$ incorrect $w_3(4) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 02

$z_6^e = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \rightarrow \left.\begin{matrix} v_1 = -3 \rightarrow o_1 = -1 \\ v_2 = -1 \rightarrow o_2 = -1 \\ v_3 = 1 \rightarrow o_3 = 1 \end{matrix}\right\}$ CORRECT. 02

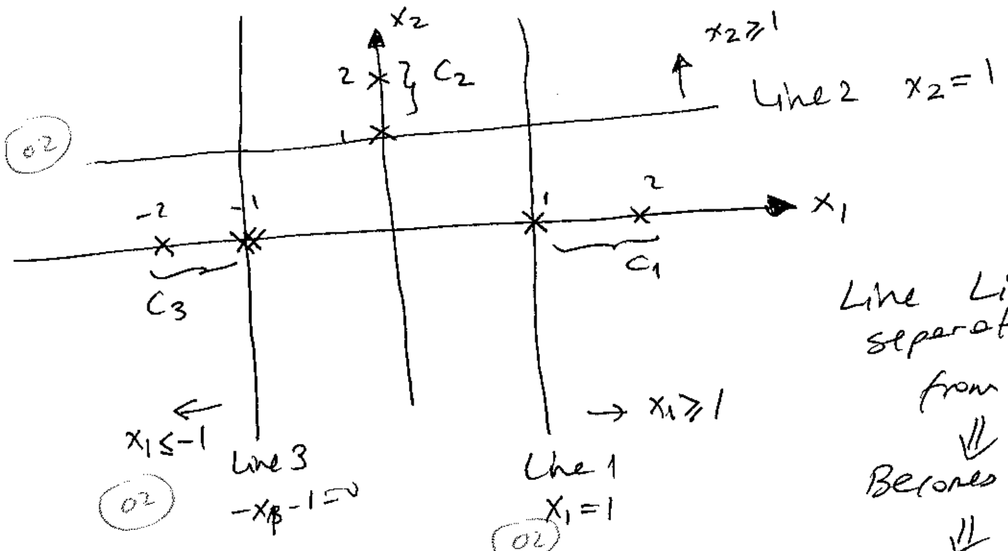
LAST UPDATED WEIGHTS : $w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $w_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $w_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 02

ii) 08

Line 1 : $x_1 - 1 = 0 \Rightarrow v_1 \geq 0$ when $x_1 \geq 1$

Line 2 : $x_2 - 1 = 0 \Rightarrow v_2 \geq 0$ when $x_2 \geq 1$

Line 3 : $-x_1 - 1 = 0 \rightarrow v_2 \geq 0$ when $x_1 \leq -1$



02

Line L_i seperates class C_i from the rest

$\Downarrow$

Becomes Linearly Separable

$\Downarrow$

CLASSIFICATION IS SOLVED!

iii) 08/08

$z_1^e = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \left.\begin{array}{l} v_1 = 0 \rightarrow o_1 = 1 \\ v_2 = -1 \rightarrow o_2 = -1 \\ v_3 = -2 \rightarrow o_3 = -1 \end{array}\right\}$ CORRECT

01

$z_2^e = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \rightarrow \left.\begin{array}{l} v_1 = 1 \rightarrow o_1 = 1 \\ v_2 = -1 \rightarrow o_2 = -1 \\ v_3 = -3 \rightarrow o_3 = -1 \end{array}\right\}$ CORRECT

01

C_1

$z_3^e = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \rightarrow \left.\begin{array}{l} v_1 = -1 \rightarrow o_1 = -1 \\ v_2 = 0 \rightarrow o_2 = 1 \\ v_3 = -1 \rightarrow o_3 = -1 \end{array}\right\}$ CORRECT

01

$z_4^e = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \rightarrow \left.\begin{array}{l} v_1 = -1 \rightarrow o_1 = -1 \\ v_2 = 1 \rightarrow o_2 = +1 \\ v_3 = -1 \rightarrow o_3 = -1 \end{array}\right\}$ CORRECT

01

C_2

$z_5^e = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \rightarrow \left.\begin{array}{l} v_1 = -2 \rightarrow o_1 = -1 \\ v_2 = -1 \rightarrow o_2 = -1 \\ v_3 = 0 \rightarrow o_3 = 1 \end{array}\right\}$ CORRECT

01

$z_6^e = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \rightarrow \left.\begin{array}{l} v_1 = -3 \rightarrow o_1 = -1 \\ v_2 = -1 \rightarrow o_2 = -1 \\ v_3 = 1 \rightarrow o_3 = 1 \end{array}\right\}$ CORRECT

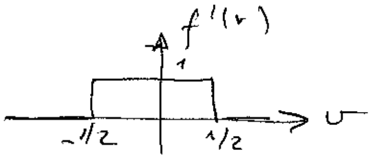
01

C_3

~~Does~~ the classification is done correctly.

02

PROB. 4: i) $f'(v) = \begin{cases} 0 & |v| > 1/2 \\ 1 & |v| \le 1/2 \end{cases}$



12/12

$$w(1) = \begin{pmatrix} 0 \\ -1.5 \\ 0.25 \end{pmatrix}$$

$z_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rightarrow v = -0.25 \Rightarrow o = 0.25 \Rightarrow e = d - o = 0.75$ 03

$$w(2) = \begin{pmatrix} 0 \\ -1.5 \\ 0.25 \end{pmatrix} + 1.1.\ 0.75 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.75 \\ -1.5 \\ -0.5 \end{pmatrix}$$ 03

$z_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \rightarrow v = w(2)^T z_2 = 0.75 - 1.5 + 0.5 = -0.25 \Rightarrow o = 0.25 \Rightarrow e = d - o = 0 - 0.25 = -0.25$ 03

$$w(3) = \begin{pmatrix} 0.75 \\ -1.5 \\ -0.5 \end{pmatrix} + 1.1.\ (-0.25) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.50 \\ -1.75 \\ -0.25 \end{pmatrix}$$ 03

ii) $w = \begin{pmatrix} 0.50 \\ -1.75 \\ -0.25 \end{pmatrix}$ 03

08/03

$z_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rightarrow v = 0.75 \Rightarrow o = 1 \Rightarrow e_1 = d - o = 1 - 1 = 0$ 03

$z_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \rightarrow v = 0.50 - 1.75 + 0.25 = -1 \Rightarrow o = 0 \Rightarrow e_2 = d - o = 0 - 0 = 0$

$$E_{ave} = \frac{1}{2}\sqrt{0^2 + 0^2} = 0$$ 02