Solutions to HW #10

① $\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \Rightarrow \begin{matrix} V_1 = AV_2 + BI_2 \\ I_1 = CV_2 + DI_2 \end{matrix}$

Fig. A

$A = \frac{V_1}{V_2}\Big|_{I_2=0} \Rightarrow V_2 = \frac{V_1}{Z_1+Z_3} Z_3 \Rightarrow \frac{V_1}{V_2} = \frac{Z_1+Z_3}{Z_3} \Rightarrow \boxed{A = 1 + Z_1/Z_3}$

$B = \frac{V_1}{I_2}\Big|_{V_2=0} \Rightarrow I_2 = \frac{V_1}{Z_1 + Z_2 /\!/ Z_3} Z_2 /\!/ Z_3 \frac{1}{Z_2} \Rightarrow I_2 = \frac{V_1}{Z_1 + \frac{Z_2Z_3}{Z_2+Z_3}} \frac{Z_2Z_3}{Z_2+Z_3} \cdot \frac{1}{Z_2} = \frac{V_1 Z_3}{(Z_1Z_2 + Z_1Z_3 + Z_2Z_3)}$

$\Rightarrow \frac{V_1}{I_2} = \frac{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}{Z_3} \Rightarrow \boxed{B = Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3}}$

$\boxed{C = \frac{I_1}{V_2}\Big|_{I_2=0} = \frac{1}{Z_3}}$

$D = \frac{I_1}{I_2}\Big|_{V_2=0} \Rightarrow I_1\Big|_{V_2=0} = \frac{V_1}{Z_1 + \frac{Z_2Z_3}{Z_2+Z_3}}, \quad I_2\Big|_{V_2=0} = \frac{V_1}{Z_1 + \frac{Z_2Z_3}{Z_2+Z_3}} \frac{Z_2Z_3}{Z_2+Z_3} \frac{1}{Z_2}$

$\Rightarrow \frac{I_1}{I_2}\Big|_{V_2=0} = \frac{\cancel{V_1}}{\cancel{Z_1 + \frac{Z_2Z_3}{Z_2+Z_3}}} \cdot \frac{\cancel{Z_1 + \frac{Z_2Z_3}{(Z_2+Z_3)}}}{\cancel{V_1}} \frac{Z_2+Z_3}{Z_2Z_3} Z_2 = \frac{Z_2+Z_3}{Z_3}$

$\Rightarrow \boxed{D = 1 + Z_2/Z_3}$

Fig. B

$A = \frac{V_1}{V_2}\Big|_{I_2=0}$ let I_x be the current passing through Y_3 & Y_2 $\Rightarrow V_1 = I_x\left(\frac{1}{Y_3} + \frac{1}{Y_2}\right)$

$V_2 = I_x / Y_2$

$\Rightarrow \frac{V_1}{V_2}\Big|_{I_2=0} = \frac{1/Y_2 + 1/Y_3}{1/Y_2} = \frac{Y_2+Y_3}{Y_3} \Rightarrow \boxed{A = 1 + \frac{Y_2}{Y_3}}$

$B = \frac{V_1}{I_2}\Big|_{V_2=0} \Rightarrow V_1 = \frac{I_2}{Y_3} \Rightarrow \boxed{B = \frac{1}{Y_3}}$

$C = \frac{I_1}{V_2}\Big|_{I_2=0} \Rightarrow \frac{I_1}{V_1} = Y_1 + \frac{Y_2Y_3}{Y_2+Y_3} \quad (*)$ also $V_2 = V_1\left(\frac{Y_3}{Y_2+Y_3}\right) \Rightarrow V_1 = \frac{V_2(Y_2+Y_3)}{Y_3} \quad (**)$

from (*) & (**) $I_1 = \left(Y_1 + \frac{Y_2Y_3}{Y_2+Y_3}\right)\left(\frac{Y_2+Y_3}{Y_3}\right)V_2 \Rightarrow \frac{I_1}{V_2}\Big|_{I_2=0} = \frac{Y_1Y_2 + Y_1Y_3 + Y_2Y_3}{Y_3}$

$\Rightarrow \boxed{C = Y_1 + Y_2 + \frac{Y_1Y_2}{Y_3}}$

$D = \frac{I_1}{I_2}\Big|_{V_2=0} \quad \frac{I_1 - I_2}{Y_1} = \frac{I_2}{Y_3} \Rightarrow I_1 = I_2 + I_2\frac{Y_1}{Y_3} \Rightarrow \boxed{D = 1 + \frac{Y_1}{Y_3}}$

Note that for both figures, you can find $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ as the cascade connection of 3 2-port networks! Can you verify it?

(2) $\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \Rightarrow A = \frac{V_1}{V_2}\Big|_{I_2=0} \Rightarrow$ port 2 is o.c.

$$\left. \begin{aligned} V_1\Big|_{z=-l} &= V^+ e^{j\beta l} + V^- e^{-j\beta l} \\ V_2\Big|_{z=0} &= V^+ + V^- \end{aligned} \right\} \quad \frac{V_1}{V_2} = \frac{V^+ e^{j\beta l} + V^- e^{-j\beta l}}{V^+ + V^-}$$

but port 2 is o.c. $\Rightarrow \Gamma = 1 \Rightarrow V^+ = V^-$

$$\Rightarrow A = \frac{V_1}{V_2}\Big|_{I_2=0} = \frac{V^+\left(e^{j\beta l} + e^{-j\beta l}\right)}{2V^+} = \cos\beta l \Rightarrow \boxed{A = \cos\beta l}$$

$$B = \frac{V_1}{I_2}\Big|_{V_2=0} \Rightarrow \text{port 2 is s.c.} = \frac{V^+ e^{j\beta l} + V^- e^{-j\beta l}}{\frac{V^+}{Z_0} - \frac{V^-}{Z_0}}$$

but $V^+ = -V^- \Rightarrow B = \dfrac{V^+\left(e^{j\beta l} - e^{-j\beta l}\right)}{2V^+/Z_0}$

$$\Rightarrow \boxed{B = j Z_0 \sin\beta l}$$

$$C = \frac{I_1}{V_2}\Big|_{I_2=0} = \frac{V^+/Z_0\left(e^{j\beta l} - e^{-j\beta l}\right)}{2V^+} = \boxed{jY_0 \sin\beta l = C}$$

$$D = \frac{I_1}{I_2}\Big|_{V_2=0} = \frac{V^+/Z_0\, e^{j\beta l} - V^-/Z_0\, e^{-j\beta l}}{\frac{V^+}{Z_0} - \frac{V^-}{Z_0}}$$

but $V^+ = -V^- \Rightarrow D = \dfrac{I_1}{I_2}\Big|_{V_2=0} = \dfrac{\frac{V^+}{Z_0}\left(e^{j\beta l} + e^{-j\beta l}\right)}{\frac{V^+}{Z_0}\, 2}$

$$\Rightarrow \boxed{D = \cos\beta l}$$

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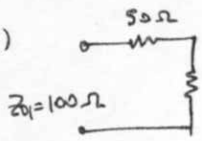
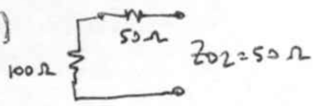
(3) (a)



$S_{11} = \Gamma_{in} = \dfrac{100 - Z_{01}}{100 + Z_{01}}$ but $Z_{01} = 100\,\Omega \Rightarrow \boxed{S_{11} = 0}$

(b)



$S_{22} = \Gamma_{out} = \dfrac{150 - Z_{02}}{150 + Z_{02}} = \dfrac{100}{200} \Rightarrow \boxed{S_{22} = 1/2}$

(c) $S_{21} = \dfrac{b_2}{a_1}\Big|_{a_2=0}$ $\quad I_1 = -I_2 = \dfrac{1}{\sqrt{Z_{01}}}(a_1 - \cancel{b_1}^{\,0}) = -\dfrac{1}{\sqrt{Z_{02}}}(\cancel{a_2}^{\,0} - b_2)$ $\quad a_2 = 0$ (for S_{21}), $b_1 = 0$ since $S_{11} = 0$

$$\Rightarrow S_{21} = \frac{b_2}{a_1} = \sqrt{\frac{Z_{02}}{Z_{01}}} = \sqrt{\frac{50}{100}} = \frac{1}{\sqrt{2}}$$

(d) $S_{12} = S_{21}$ due to reciprocity (we have a simple resistor)

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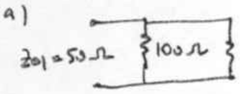
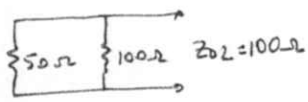
(4) (a)



$\Rightarrow \boxed{S_{11} = 0}$

(b)



$50 // 100 = \dfrac{100}{3}$

$S_{22} = \dfrac{100/3 - 100}{100/3 + 100} \Rightarrow \boxed{S_{22} = -\frac{1}{2}}$

(c) $S_{21} = \dfrac{b_2}{a_1}\Big|_{a_2=0}$ $\quad V_1 = V_2 \Rightarrow \sqrt{50}\,(a_1 + \cancel{b_1}^{\,0}) = \sqrt{100}\,(\cancel{a_2}^{\,0} + b_2)$ since $S_{11} = 0$

$$\Rightarrow S_{21} = \frac{b_2}{a_1}\Big|_{a_2=0} = \sqrt{\frac{50}{100}} = \frac{1}{\sqrt{2}}$$

(d) $S_{12} = S_{21}$ since device is reciprocal

5. $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ $[S]$ is reciprocal $\Rightarrow S_{12} = S_{21}$, $[S]$ is unitary $\Rightarrow$ network is lossless

$$\left.\begin{array}{ll} |S_{11}|^2 + |S_{12}|^2 = 1 & \text{——— (a)} \\ S_{11} S_{12}^* + S_{12} S_{22}^* = 0 & \text{——— (b)} \\ |S_{12}|^2 + |S_{22}|^2 = 1 & \text{——— (c)} \end{array}\right\}$$ from (a) & (c) $|S_{11}|^2 = |S_{22}|^2 \Rightarrow |S_{11}| = |S_{22}|$

then $\boxed{|S_{12}| = \sqrt{1 - |S_{11}|^2}}$

$$\left.\begin{array}{l} S_{11} = |S_{11}| e^{j\theta_1} \\ S_{22} = |S_{22}| e^{j\theta_2} = |S_{11}| e^{j\theta_1} \\ S_{12} = S_{21} = |S_{12}| e^{j\phi_1} = (1 - |S_{11}|^2)^{1/2} e^{j\phi_1} \end{array}\right\}$$ use these eqns. together with eqn. (b)

$$\Rightarrow |S_{11}| e^{j\theta_1} (1 - |S_{11}|^2)^{1/2} e^{-j\phi_1} + (1 - |S_{11}|^2)^{1/2} e^{j\phi_1} |S_{11}| e^{-j\theta_2} = 0$$

$$\Rightarrow \underbrace{|S_{11}| \left[1 - |S_{11}|^2\right]^{1/2}}_{\neq 0} \underbrace{\left[e^{j\theta_1} e^{-j\phi_1} + e^{j\phi_1} e^{-j\theta_2}\right]}_{=0} = 0 \Rightarrow e^{j\theta_1} e^{-j\phi_1} = -e^{j\phi_1} e^{-j\theta_2} \Rightarrow e^{j(\theta_2 + \theta_1)} = -e^{j2\phi_1}$$

$$\Rightarrow \boxed{\phi_1 = \frac{\theta_1 + \theta_2}{2} + \frac{\pi}{2} + n\pi}$$

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6. (a) If all ports are matched, there is no reflection $\Rightarrow S_{ii} = 0$ for $i = 1, 2, 3$

(b) If network is reciprocal, then $S_{ij} = S_{ji} \Rightarrow [S] = \begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{12} & 0 & S_{23} \\ S_{13} & S_{23} & 0 \end{bmatrix}$ and if the network is lossless, then $[S]$ matrix is unitary.

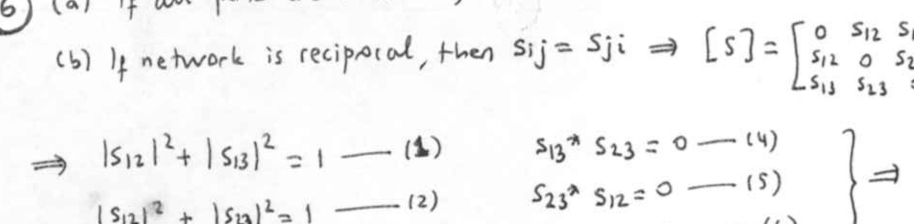
$$\Rightarrow \begin{array}{ll} |S_{12}|^2 + |S_{13}|^2 = 1 \text{ —— (1)} & S_{13}^* S_{23} = 0 \text{ —— (4)} \\ |S_{12}|^2 + |S_{23}|^2 = 1 \text{ —— (2)} & S_{23}^* S_{12} = 0 \text{ —— (5)} \\ |S_{13}|^2 + |S_{23}|^2 = 1 \text{ —— (3)} & S_{12}^* S_{13} = 0 \text{ —— (6)} \end{array}$$ $\Rightarrow$ From (4) – (6) at least two of the three parameters (S_{12}, S_{13}, S_{23}) must be zero.

However, this condition is inconsistent with equations (a) – (c) implying that a three-port network can not be lossless, reciprocal and matched at all ports.

(c) $[S] = \begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{21} & 0 & S_{23} \\ S_{31} & S_{32} & 0 \end{bmatrix}$, network is lossless $\Rightarrow [S]$ is unitary $\Rightarrow$ $\begin{array}{l} |S_{12}|^2 + |S_{32}|^2 = 1 \\ |S_{21}|^2 + |S_{31}|^2 = 1 \\ |S_{13}|^2 + |S_{23}|^2 = 1 \end{array}$ & $\begin{array}{l} S_{31}^* S_{32} = 0 \\ S_{21}^* S_{23} = 0 \\ S_{12}^* S_{13} = 0 \end{array}$

$$\Rightarrow S_{32} = 1, \; S_{13} = 1, \; S_{21} = 1 \Rightarrow [S] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ $[S]$ matrix of a lossless circulator which works in the clockwise direction.

7.



$V_1^- = S_{11}^{(1)} V_1^+ + S_{12}^{(1)} V_2^+$ but $V_2^+ = \Gamma_{in}^{(2)} V_2^-$

$V_2^- = S_{21}^{(1)} V_1^+ + S_{22}^{(1)} V_2^+$ $\Rightarrow V_2^+ = S_{11}^{(2)} V_2^-$

$$\left.\begin{array}{l} \Rightarrow V_1^- = S_{11}^{(1)} V_1^+ + S_{12}^{(1)} S_{11}^{(2)} V_2^- \\ V_2^- = S_{21}^{(1)} V_1^+ + S_{22}^{(1)} S_{11}^{(2)} V_2^- \Rightarrow V_2^- = \dfrac{S_{21}^{(1)} V_1^+}{1 - S_{22}^{(1)} S_{11}^{(2)}} \end{array}\right\} \Rightarrow V_1^- = S_{11}^{(1)} V_1^+ + S_{12}^{(1)} S_{11}^{(2)} \left[\frac{S_{21}^{(1)} V_1^+}{1 - S_{22}^{(1)} S_{11}^{(2)}}\right]$$

$$\Rightarrow V_1^- = V_1^+ \left[S_{11}^{(1)} + S_{11}^{(2)} \frac{S_{12}^{(1)} S_{21}^{(1)}}{1 - S_{11}^{(2)} S_{22}^{(1)}}\right] \Rightarrow \boxed{\Gamma_{in} = \frac{V_1^-}{V_1^+} = S_{11}^{(1)} + S_{11}^{(2)} \frac{S_{12}^{(1)} S_{21}^{(1)}}{1 - S_{11}^{(2)} S_{22}^{(1)}}}$$