Solutions to HW #2

1-)

$$\nabla \times \bar{E} = -j\omega\mu\bar{H}$$
$$\nabla \times \bar{H} = \bar{J}_i + \sigma\bar{E} + j\omega\varepsilon\bar{E}$$

$$\bar{H}^* \cdot (\nabla \times \bar{E}) = -j\omega\mu \,|\, \bar{H} \,|^2$$
$$\bar{E} \cdot (\nabla \times \bar{H}^*) = \bar{E} \cdot \bar{J}_i^* + \sigma \,|\, \bar{E} \,|^2 - j\omega\varepsilon^* \,|\, \bar{E} \,|^2$$

Use vector identity;

$$\nabla \cdot (\bar{A} \times \bar{B}) = \left(\nabla \times \bar{A}\right) \cdot \bar{B} - (\nabla \times \bar{B}) \cdot \bar{A}$$
$$\nabla \cdot (\bar{E} \times \bar{H}^*) = \left(\nabla \times \bar{E}\right) \cdot \bar{H}^* - (\nabla \times \bar{H}^*) \cdot \bar{E}$$
$$\nabla \cdot (\bar{E} \times \bar{H}^*) = -j\omega\mu \,|\, \bar{H} \,|^2 - (\bar{E} \cdot \bar{J}_i^* + \sigma \,|\, \bar{E} \,|^2 - j\omega\varepsilon^* \,|\, \bar{E} \,|^2)$$

$$\varepsilon = \varepsilon' - j\varepsilon'' \rightarrow \varepsilon^* = \varepsilon' + j\varepsilon''$$
$$\mu = \mu' - j\mu''$$

$$\begin{aligned}\nabla \cdot (\bar{E} \times \bar{H}^*) &= -j\omega\left(\mu' - j\mu''\right) |\, \bar{H} \,|^2 - (\bar{E} \cdot \bar{J}_i^* + \sigma \,|\, \bar{E} \,|^2 - j\omega(\varepsilon' + j\varepsilon'') \,|\, \bar{E} \,|^2) \\ &= -j\omega\mu' \,|\, \bar{H} \,|^2 - \omega\mu'' \,|\, \bar{H} \,|^2 - \bar{E} \cdot \bar{J}_i^* - \sigma \,|\, \bar{E} \,|^2 + j\omega\varepsilon' \,|\, \bar{E} \,|^2 - \omega\varepsilon'' \,|\, \bar{E} \,|^2\end{aligned}$$

Now integrate over volume,

$$\int_V \nabla \cdot (\bar{E} \times \bar{H}^*) dV = \int_V \left(-j\omega\mu' \,|\, \bar{H} \,|^2 - \omega\mu'' \,|\, \bar{H} \,|^2 - \bar{E} \cdot \bar{J}_i^* - \sigma \,|\, \bar{E} \,|^2 + j\omega\varepsilon' \,|\, \bar{E} \,|^2 - \omega\varepsilon'' \,|\, \bar{E} \,|^2\right) dV$$

We can apply Divergence's theorem to LHS.

$$\begin{aligned}\int_S (\bar{E} \times \bar{H}^*) \cdot d\bar{S} = {} & \int_V -j\omega\mu' \,|\, \bar{H} \,|^2 dV - \int_V \omega\mu'' \,|\, \bar{H} \,|^2 dV - \int_V \bar{E} \cdot \bar{J}_i^* dV \\ & - \int_V \sigma \,|\, \bar{E} \,|^2 dV + \int_V j\omega\varepsilon' \,|\, \bar{E} \,|^2 dV - \int_V \omega\varepsilon'' \,|\, \bar{E} \,|^2 dV\end{aligned}$$

$$\begin{aligned}\int_S (\bar{E} \times \bar{H}^*) \cdot d\bar{S} = {} & -\int_V \bar{E} \cdot \bar{J}_i^* dV - \int_V \sigma \,|\, \bar{E} \,|^2 dV + j\omega\left(\int_V \varepsilon' \,|\, \bar{E} \,|^2 dV - \int_V \mu' \,|\, \bar{H} \,|^2 dV\right) \\ & - \omega\left(\int_V \varepsilon'' \,|\, \bar{E} \,|^2 dV + \int_V \mu'' \,|\, \bar{H} \,|^2 dV\right)\end{aligned}$$

Multiplying both side with $\frac{1}{2}$ and rearranging the equation,

$$\boxed{\begin{aligned}-\frac{1}{2}\int_V \bar{E}\cdot\bar{J}_i^* dV &= \frac{1}{2}\int_S (\bar{E}\times\bar{H}^*)\cdot d\bar{S} + \frac{\sigma}{2}\int_V |\bar{E}|^2 dV + \frac{j\omega}{2}\left(\int_V (\mu' |\bar{H}|^2 - \varepsilon' |\bar{E}|^2) dV\right)\\ &+ \frac{\omega}{2}\left(\int_V (\varepsilon'' |\bar{E}|^2 + \mu'' |\bar{H}|^2) dV\right)\end{aligned}}$$

Solutions to HW # 3

(2) For a parallel plate TLIN

$C = \epsilon \frac{W}{d}$ (F/m)

$L = \mu \frac{d}{W}$ (H/m)

and when the line is lossless $Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu d/w}{\epsilon w/d}}$

$\Rightarrow Z_0 = \frac{d}{w}\sqrt{\mu/\epsilon}$

(i) Z_0 is to be kept constant when $\epsilon_r' = 2\epsilon_r \Rightarrow d' = ?$

$\Rightarrow \frac{d}{w}\sqrt{\frac{\mu}{\epsilon}} = \frac{d'}{w}\sqrt{\frac{\mu}{2\epsilon}} \Rightarrow \underline{d' = \sqrt{2}\, d}$

(ii) Z_0 is to be kept constant when $\epsilon_r' = 2\epsilon_r \Rightarrow w' = ?$

$\frac{d}{w'}\sqrt{\frac{\mu}{2\epsilon}} = \frac{d}{w}\sqrt{\frac{\mu}{\epsilon}} \Rightarrow \underline{w' = \frac{w}{\sqrt{2}}}$

(iii) Z_0 is to be kept constant when $d' = 2d \Rightarrow w' = ?$

$\frac{d}{w}\sqrt{\frac{\mu}{\epsilon}} = \frac{2d}{w'}\sqrt{\frac{\mu}{\epsilon}} \Rightarrow \underline{w' = 2w}$

(iv) $v_p = \frac{1}{\sqrt{\mu\epsilon}} \Rightarrow v_{p_i} = v_p/\sqrt{2}$ for part (i)

$v_{p_{ii}} = v_p/\sqrt{2}$ for part (ii)

$v_{p_{iii}} = v_p$ for part (iii)

---

(3) (a) lossless line $\Rightarrow Z_0 = \sqrt{\frac{L}{C}} = \left(\frac{\mu/\pi \cosh^{-1}(D/2a)}{\frac{\pi\epsilon}{\cosh^{-1}(D/2a)}}\right)^{1/2} = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon}}\cosh^{-1}\left(\frac{D}{2a}\right)$

for a two-wire TLIN

$\Rightarrow 300 = \frac{1}{\pi}\frac{120\pi}{\sqrt{2.25}}\cosh^{-1}\left(\frac{D}{2\times 6\times 10^{-4}\,m}\right)$

using the identity $\ln\left[x + \sqrt{x^2 - 1}\right] = \cosh^{-1} x$

$\Rightarrow 300 = \frac{120}{1.5}\ln\left[\frac{D}{2\times 6\times 10^{-4}} + \sqrt{\left(\frac{D}{2\times 6\times 10^{-4}}\right)^2 - 1}\right]$

$\Rightarrow D = 25.5\times 10^{-3}$ (m)

$\Rightarrow \underline{D = 25.5\text{ mm}}$

(b) for a lossless coaxial cable $Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu/2\pi \ln(b/a)}{2\pi\epsilon/\ln(b/a)}} = \frac{1}{2\pi}\ln(b/a)\sqrt{\frac{\mu}{\epsilon}}$

$\Rightarrow 75 = \frac{120\pi}{2\pi}\frac{1}{\sqrt{2.25}}\ln\left(\frac{b}{6\times10^{-4}}\right) \Longrightarrow b = 3.91\times10^{-3}\ (m) \quad (b = 3.91mm)$

---

4 (a) $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$, $Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$ $\frac{R}{L} = \frac{G}{C}$ given

$$\gamma = \sqrt{j\omega L\left(1+\frac{R}{j\omega L}\right)j\omega C\left(1+\frac{G}{j\omega C}\right)} = j\omega\sqrt{LC}\sqrt{1-j\left(\frac{R}{\omega L}+\frac{G}{\omega C}\right)-\frac{RG}{\omega^2 LC}}$$

$$= j\omega\sqrt{LC}\sqrt{1-2j\frac{R}{\omega L}-\frac{R^2}{\omega^2L^2}} = j\omega\sqrt{LC}\sqrt{\left(1-j\frac{R}{\omega L}\right)^2} = j\omega\sqrt{LC}\left(1-j\frac{R}{\omega L}\right)$$

$$\Rightarrow \gamma = \alpha + j\beta = R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC} \Longrightarrow \boxed{\alpha = R\sqrt{\frac{C}{L}},\ \beta = \omega\sqrt{LC}}$$

$$Z_0 = \sqrt{\frac{j\omega L(1+R/j\omega L)}{j\omega C(1+G/j\omega C)}} \quad \text{if } \frac{R}{L} = \frac{G}{C} \Rightarrow Z_0 = \sqrt{\frac{L}{C}} \Longrightarrow \boxed{\begin{array}{l} R_0 = \sqrt{L/C} \\ X_0 = 0 \end{array}}$$

$\beta = \omega\sqrt{LC}$ a linear function of frequency
$\alpha = R\sqrt{\frac{C}{L}}$ not a function of frequency
} distortionless line

(b) $Z_0 = 50\Omega \Longrightarrow R_0 = 50\Omega$ & $X_0 = 0$
$\alpha = 0.01\ (dB/m)$
$\beta = 0.8\pi\ (rad/m)$
$f = 100\ MHz$

} This is a distortionless line with $\alpha = R\sqrt{\frac{C}{L}}$, $\beta = \omega\sqrt{LC}$ and $Z_0 = \sqrt{\frac{L}{C}}$

$\alpha = 0.01\,dB/m \simeq 0.00115\ Np/m \Longrightarrow R = \alpha\sqrt{\frac{L}{C}} = \alpha Z_0 \simeq 0.0575\ (\Omega/m)$

$G = \frac{C}{L}R = \frac{\alpha^2}{R} = 2.3\times10^{-5} = 23\mu S/m$

$\beta Z_0 = \omega\sqrt{LC}\sqrt{\frac{L}{C}} \Rightarrow \beta Z_0 = \omega L \Rightarrow L = \frac{\beta Z_0}{\omega} = 200nH/m$

$Z_0 = \sqrt{\frac{L}{C}} \Rightarrow C = \frac{L}{Z_0^2} = \frac{20\times10^{-9}}{2500} = 80pF/m$