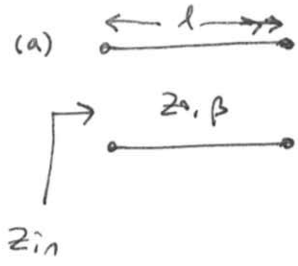
Solutions to HW # 4

1 (a)



$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \quad \text{if } Z_L \to \infty$$

$$\Rightarrow Z_{in} = -jZ_0 \cot(\beta l) = -j \frac{Z_0}{\tan(\beta l)}$$

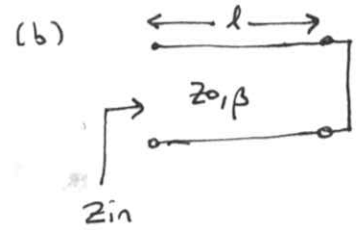
if $\beta l \ll 1 \Rightarrow \tan(\beta l) \approx \beta l$

$$\Rightarrow Z_{in} \simeq -j \frac{Z_0}{\beta l} \quad \text{use} \quad Z_0 = \sqrt{\frac{L}{C}} \quad \text{and} \quad \beta = \omega\sqrt{LC}$$

$$\Rightarrow Z_{in} \simeq -j \sqrt{\frac{L}{C}} \frac{1}{\omega\sqrt{LC}} \frac{1}{l} \Rightarrow -j \frac{1}{\omega C l} \;//$$

where Cl equivalent impedance

(b)



$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \quad \text{if } Z_L \to 0$$

$$\Rightarrow Z_{in} = jZ_0 \tan(\beta l) \quad \text{if} \quad \beta l \ll 1 \Rightarrow \tan(\beta l) \approx \beta l$$

$$\Rightarrow Z_{in} \simeq jZ_0 \beta l = j\sqrt{\frac{L}{C}}\, \omega\sqrt{LC}\, l = j\omega L l \;//$$

where Ll: equivalent inductance

2

$$\left.\begin{aligned} Z_{in}^{oc} &= -jZ_0 \cot(\beta l) \\ Z_{in}^{sc} &= jZ_0 \tan(\beta l) \end{aligned}\right\} \Rightarrow Z_{in}^{oc} Z_{in}^{sc} = Z_0^2 \Rightarrow \boxed{Z_0 = \sqrt{Z_{in}^{oc} Z_{in}^{sc}}}$$

$$\Rightarrow \frac{Z_{in}^{sc}}{Z_{in}^{oc}} = \frac{jZ_0 \tan(\beta l)}{-jZ_0 \cot(\beta l)} = -\tan^2(\beta l) \Rightarrow \tan(\beta l) = \sqrt{-\frac{Z_{in}^{sc}}{Z_{in}^{oc}}}$$

$$\Rightarrow \boxed{\beta = \frac{1}{l} \tan^{-1} \sqrt{\frac{-Z_{in}^{sc}}{Z_{in}^{oc}}}}$$

3 $V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z} \Rightarrow V(z) = V^+ \left(e^{-j\beta z} + \Gamma_L e^{j\beta z} \right)$ with $\Gamma_L = \dfrac{R_L - \sqrt{Z_0 R_L}}{R_L + \sqrt{Z_0 R_L}}$

$$V(z=-l) = V^+ \left(e^{j\beta l} + \Gamma_L e^{-j\beta l} \right) = V_i$$

$$\Rightarrow \boxed{V^+ = \frac{V_i}{\left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right)}} \Longrightarrow \boxed{V^- = \Gamma_L \frac{V_i}{\left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right)}}$$

OR (second method)

$$V(z) = V^+ e^{-j\beta(z+l)} + V^- e^{j\beta(z+l)}$$

$V(z=-l) = V_i = V^+ + V^-$, $V(z=0) = V^+ e^{-j\beta l} + V^- e^{j\beta l}$

$$\Gamma_L = \frac{V^- e^{j\beta l}}{V^+ e^{-j\beta l}} \Rightarrow V^- = \Gamma_L V^+ e^{-j\beta l 2}$$

$$V_i = V^+ + \Gamma_L V^+ e^{-j\beta l 2} = V^+ \left(1 + \Gamma_L e^{-j2\beta l} \right) \Rightarrow \boxed{V^+ = \frac{V_i}{1 + \Gamma_L e^{-j2\beta l}}}$$

$$\Rightarrow \boxed{V^- = \frac{\Gamma_L e^{-j2\beta l} V_i}{1 + \Gamma_L e^{-j2\beta l}}}$$

Note that even if the V^+ & V^- expressions found using a different reference (second method) are different than the first method, notice that $V(z)$ expression is same in both methods

1st way: $V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z} \Rightarrow \boxed{V(z) = \dfrac{V_i \; e^{-j\beta z}}{\left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right)} + \Gamma_L \dfrac{V_i \, e^{j\beta z}}{\left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right)}}$

2nd way: $V(z) = V^+ e^{-j\beta(z+l)} + V^- e^{j\beta(z+l)}$

$$\Rightarrow V(z) = \frac{V_i \; e^{-j\beta z} e^{-j\beta l}}{\left(1 + \Gamma_L e^{-j2\beta l}\right)} + \frac{\Gamma_L e^{-j2\beta l} V_i}{\left(1 + \Gamma_L e^{-j\beta l 2}\right)} e^{j\beta z} e^{j\beta l}$$

same

$$\Rightarrow \boxed{V(z) = \frac{V_i e^{-j\beta z}}{\left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right)} + \Gamma_L \frac{V_i \, e^{j\beta z}}{\left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right)}}$$

Solutions to HW #4

4 $f = 1\,GHz \Rightarrow \lambda_0 = \frac{c}{f} = 30\,cm \Rightarrow l_1 = 7.5\,cm \Rightarrow l_1 = \frac{\lambda_0}{4}$ QWT

$\Rightarrow \lambda_1 = \frac{c/\sqrt{\epsilon_r}}{f} = 15\,cm \Rightarrow l_2 = 7.5\,cm \Rightarrow l_2 = \frac{\lambda}{2}$ Half wave transformer

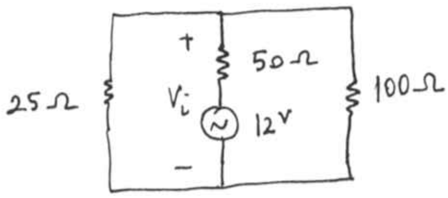
$SWR = 2 = \frac{1+|\Gamma|}{1-|\Gamma|} \Rightarrow |\Gamma| = \frac{1}{3}$ $\quad$ ($Z_{Li} > Z_0$ & Z_{Li} are real for $i = 1, 2$)

$\Rightarrow \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1}{3} \Rightarrow Z_L = 100\,\Omega$

For l_1 if $Z_{L1} = 100\,\Omega \Rightarrow Z_{in1} = \frac{Z_0^2}{Z_{L1}} = \frac{50 \times 50}{100} = 25\,\Omega$ (QWT)

For l_2 if $Z_{L2} = 100\,\Omega \Rightarrow Z_{in2} = Z_{L2} = 100\,\Omega$ (Half-wave transformer)

Then, the equivalent cct. becomes



$\Rightarrow V_i = \frac{12^V}{50 + 100//25} \cdot 25//100 = \frac{24}{7}\,V$

$P_1 = \frac{1}{2}\frac{|V_i|^2}{R_1} = \frac{1}{2}\left(\frac{24}{7}\right)^2 \frac{1}{25} = \frac{0.47}{2}$ Watts (On the quarter wave section)

$P_2 = \frac{1}{2}\frac{|V_i|^2}{R_2} = \frac{1}{2}\left(\frac{24}{7}\right)\frac{1}{100} = \frac{0.12}{2}$ watts (On the half-wave section)

---

5 $V_i = V_0^+\left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right)\Big|_{z=-l} = V_0^+ e^{-j\beta z}\left(1 + \Gamma_L e^{j2\beta z}\right)\Big|_{z=-l}$

where $V_0^+ = V_g \frac{Z_0}{Z_0 + Z_g} \frac{e^{-j\beta l}}{\left[1 - \Gamma_L \Gamma_g e^{-j2\beta l}\right]}$ $\quad$ with $\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = 0$, $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0.447\angle 0.648\pi$

$\beta = \frac{\omega}{c} = \frac{2\pi \times 10^8}{3 \times 10^8} = \frac{2\pi}{3}\,rad/m \Rightarrow \beta l = 2.4\pi$

$\Rightarrow V_i = V_g \frac{Z_0}{Z_0 + Z_g} e^{-j\beta l} e^{j\beta l}\left(1 + \Gamma_L e^{-j2\beta l}\right) = 10\frac{50}{100} e^{-j2.4\pi} e^{j2.4\pi}\left(1 + 0.447 e^{j0.648\pi} e^{-j4.8\pi}\right)$

$\Rightarrow \boxed{V_i = 7.06\angle{-8.43}\ (V)}$

(b) $V_L = V(0) = V_0^+(1 + \Gamma_L) = V_g \frac{Z_0}{Z_0 + Z_g} e^{-j\beta l}(1 + \Gamma_L) = 10\frac{50}{100} e^{-j2.4\pi}\left(1 + 0.447 e^{j0.648\pi}\right)$

$\Rightarrow \boxed{V_L = 4.47\angle{-45.5^\circ}\ (V)}$

(d) $P_{ave} = \frac{1}{2}\left|\frac{V_L}{Z_L}\right|^2 R_L = \frac{1}{2}\left(\frac{4.47}{35.36}\right)^2 25 = \underline{0.2\,W}$

(c) $VSWR = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+0.447}{1-0.447} = 2.62$

6 (a) $\alpha = 1.5\ dB/m \Rightarrow \alpha = \frac{1.5}{8.686}\ Np/m$ (This conversion is necessary, since the unit of α is Np/m)

$$\gamma = \alpha + j\beta \quad , \quad \beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = 20.944\ rad/m$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1}{3} \quad / \quad \Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = 0$$

$$\Rightarrow V_0^+ = V_g \frac{Z_0}{Z_0 + Z_g} \frac{e^{-\gamma l}}{\left[1 - \Gamma_L \Gamma_g e^{-2\gamma l}\right]} \Rightarrow V_0^+ = V_g \frac{Z_0}{Z_0 + Z_g} e^{-\gamma l}$$

$$\Rightarrow P_{in} = \frac{|V_0^+|^2}{2Z_0}\left[e^{2\alpha l} - |\Gamma_L|^2 e^{-2\alpha l}\right] = 0.243/_2 \text{ Watts}$$

$$\Rightarrow P_L = \frac{|V_0^+|^2}{2Z_0}\left[1 - |\Gamma_L|^2\right] = \frac{0.111}{2} \text{ Watts}$$

(b) Return loss $= 10 \log \frac{1}{|\Gamma|^2} \Rightarrow$ At the input $|\Gamma_{in}| = |\Gamma_L| e^{-2\alpha l}$

$$RL_{input} = 10 \log_{10} \frac{1}{|\Gamma_{in}|^2} = 15.542\ dB$$

$$RL_{output} = 10 \log_{10} \frac{1}{|\Gamma_L|^2} = 9.54\ dB.$$

---

7 $V_0^+ = V_g \frac{Z_{in}}{Z_{in} + Z_g} \frac{1}{(e^{j\beta l} + \Gamma_L e^{-j\beta l})}$ with $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$ and $Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$

also $Z_{in} = Z_0 \frac{1 + \Gamma_L e^{-j2\beta l}}{1 - \Gamma_L e^{-j2\beta l}}$ $\Rightarrow V_0^+ = V_g \frac{Z_0 \frac{1 + \Gamma_L e^{-j2\beta l}}{1 - \Gamma_L e^{-j2\beta l}}}{\left(Z_0 \frac{1 + \Gamma_L e^{-j2\beta l}}{1 - \Gamma_L e^{-j2\beta l}} + Z_g\right)\left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right)}$

$$\Rightarrow V_0^+ = V_g \frac{Z_0 \frac{1 + \cancel{\Gamma_L e^{-j2\beta l}}}{\cancel{1 - \Gamma_L e^{-j2\beta l}}}}{\frac{\left[Z_0 + Z_0 \Gamma_L e^{-j2\beta l} + Z_g - Z_g \Gamma_L e^{-j2\beta l}\right]}{\cancel{1 - \Gamma_L e^{-j2\beta l}}} e^{j\beta l}\left(\cancel{1 + \Gamma_L e^{-j2\beta l}}\right)} = V_g \frac{Z_0 e^{-j\beta l}}{(Z_0 + Z_g) + (Z_0 - Z_g)\Gamma_L e^{-j2\beta l}}$$

$$\Rightarrow V_0^+ = V_g \frac{Z_0 e^{-j\beta l}}{(Z_0 + Z_g)\left[1 + \frac{Z_0 - Z_g}{Z_0 + Z_g} \Gamma_L e^{-j2\beta l}\right]} \Rightarrow \boxed{V_0^+ = V_g \frac{Z_0}{Z_0 + Z_g} \frac{e^{-j\beta l}}{\left[1 - \Gamma_L \Gamma_g e^{-j2\beta l}\right]}}$$

where $\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}$