EEE 451 / 551

Solutions to HW # 7

(1) $k^2 = k_c^2 + k_z^2 \Rightarrow k_z = \sqrt{k^2 - k_c^2}$ cut-off occurs when $k^2 = k_c^2$ but $k^2 = \omega^2 \mu \epsilon = (2\pi f)^2 \mu \epsilon$

and $k_c^2 = \omega_c^2 \mu \epsilon = (2\pi f_c)^2 \mu \epsilon \Rightarrow k_z = k\sqrt{1-(k_c/k)^2} = k\sqrt{(1-(f_c/f)^2)}$

From class notes $Z_{TM} = \frac{k_z}{\omega \epsilon} \Rightarrow Z_{TM} = \frac{k\sqrt{1-(f_c/f)^2}}{\omega \epsilon} = \frac{\omega\sqrt{\mu\epsilon}}{\omega\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2} \Rightarrow \boxed{Z_{TM} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$

$Z_{TE} = \frac{\omega\mu}{k_z} \Rightarrow Z_{TE} = \frac{\omega\mu}{\omega\sqrt{\mu\epsilon}\sqrt{1-(f_c/f)^2}} \Rightarrow \boxed{Z_{TE} = \frac{\eta}{\sqrt{1-(f_c/f)^2}}}$

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(2) $f = 10\,GHz \Rightarrow \lambda = \frac{v}{f} = \frac{3\times10^{10}/\sqrt{2.25}\ cm/sec}{10\times10^9} = 2\,cm$ (unbounded wavelength)

$f_{c_{10}} = \frac{v}{2a} = \frac{2\times10^{10}\ cm/sec}{2\,(1.5)} \Rightarrow \boxed{f_{c_{10}} = 6.67\ GHz}$

(a) $k_z = \sqrt{k^2 - [\pi/a]^2} = \sqrt{\left(\frac{2\pi}{2\times10^{-2}}\right)^2 - \left(\frac{\pi}{1.5\times10^{-2}}\right)^2} \simeq 234\ \frac{rad}{m}$

(b) $\lambda_g = \frac{\lambda}{\sqrt{1-(f_c/f)^2}} = \frac{0.02}{0.745} = 0.0268\ m.$

(c) $v_p = \frac{\omega}{k_z} = \frac{v}{\sqrt{1-(f_c/f)^2}} = 2.68\times10^8\ m/sec$

(d) $Z_{TE} = \frac{\eta}{\sqrt{1-(f_c/f)^2}} = 337.4\,\Omega$

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(3) $\left.\begin{array}{l} a = 2.29\times10^{-2}\,m \\ b = 1.02\times10^{-2}\,m \end{array}\right\} \Rightarrow$ The 2 modes having the lowest cut-off frequency are TE_{10} & TE_{20}

$f_{c_{n,m}} = \frac{v}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \Rightarrow f_{c_{1,0}} = \frac{c}{2a} = 6.55\,GHz,\ f_{c_{2,0}} = \frac{c}{a} = 13.1\,GHz$

(Note that $f_{c_{0,1}} = \frac{c}{2b} > f_{c_{2,0}}$ and $f_{c_{11}} = \frac{c}{2a}\sqrt{1+(a/b)^2} > f_{c_{2,0}}$)

$\Rightarrow$ allowable operating freq. range under the specific condition is $1.25 f_{c_{1,0}} \le f \le 0.95 f_{c_{2,0}}$

$\Rightarrow 8.19\,GHz \le f \le 12.45\,GHz$

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(4) The dominant mode is $TE_{11} \Rightarrow f_{c_{11}} = \frac{p'_{11}}{2\pi a}c = \frac{1.841}{2\pi a}3\times10^8 \Rightarrow (f_c)_{TE_{1,1}} = \frac{0.0879}{a}\,GHz = 0.8\times10\,GHz$

$\Rightarrow a = 1.1\,cm$ OR diameter $= 2.2\,cm$.

From the given tables $(f_c)_{TE_{11}} = 8\,GHz$, $(f_c)_{TE_{21}} = 13.27\,GHz$, $(f_c)_{TM_{01}} = 10.45\,GHz$

all the other modes have f_c higher than $15\,GHz$.

(a) $\nabla \times \hat{E} = -j\omega\mu\hat{H}$, $\nabla \times \hat{H} = j\omega\epsilon$, $\nabla \cdot \hat{D} = 0$, $\nabla \cdot \hat{B} = 0$

(b) $\nabla \cdot \hat{D} = 0 \Rightarrow \hat{D} = -\nabla \times \hat{F} \Rightarrow \hat{E} = -\frac{1}{\epsilon}\nabla \times \hat{F}$, From $\nabla \times \hat{H} = j\omega\epsilon\hat{E} = j\omega\epsilon\left(-\frac{1}{\epsilon}\nabla \times \hat{F}\right)$

$\Rightarrow \underline{\nabla \times \hat{H} = -j\omega\nabla \times \hat{F}} \Rightarrow \nabla \times (\hat{H} + j\omega\hat{F}) = 0 \Rightarrow \hat{H} + j\omega\hat{F} = -\nabla\phi_m \Rightarrow \boxed{\hat{H} = -\nabla\phi_m - j\omega\hat{F}}$

From $\nabla \times \hat{E} = -j\omega\mu\hat{H} \Rightarrow \nabla \times \left(-\frac{1}{\epsilon}\nabla \times \hat{F}\right) = -\frac{1}{\epsilon}\nabla \times \nabla \times \hat{F} = -j\omega\mu\hat{H} \Rightarrow -\frac{1}{\epsilon}\left[\nabla(\nabla \cdot \hat{F}) - \nabla^2\hat{F}\right] = -j\omega\mu\hat{H}$

$\Rightarrow \nabla^2\hat{F} - \nabla(\nabla \cdot \hat{F}) + j\omega\mu\epsilon\hat{H} = 0 \Rightarrow \nabla^2\hat{F} - \nabla(\nabla \cdot \hat{F}) + j\omega\mu\epsilon(-\nabla\phi_m - j\omega\hat{F}) = 0$

$\Rightarrow \nabla^2\hat{F} + k^2\hat{F} - \nabla(\nabla \cdot \hat{F} + j\omega\mu\epsilon\phi_m) = 0$, Let $\boxed{\nabla \cdot \hat{F} = -j\omega\mu\epsilon\phi_m} \Rightarrow \boxed{\nabla^2\hat{F} + k^2\hat{F} = 0}$

$\Rightarrow \hat{H} = -\nabla\phi_m - j\omega\hat{F} \Rightarrow \hat{H} = -j\omega\hat{F} - \nabla\left(\frac{j\nabla \cdot \hat{F}}{\omega\mu\epsilon}\right) \Rightarrow \boxed{\hat{H} = -j\omega\hat{F} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla \cdot \hat{F})}$

(c) $\nabla \cdot \hat{B} = 0 \Rightarrow \hat{B} = \nabla \times \hat{A} \Rightarrow \nabla \times \hat{E} = -j\omega\nabla \times \hat{A} \Rightarrow \nabla \times (\hat{E} + j\omega\hat{A}) = 0 \Rightarrow \hat{E} + j\omega\hat{A} = -\nabla V \Rightarrow \hat{E} = -\nabla V - j\omega\hat{A}$

from $\nabla \times \hat{H} = j\omega\epsilon\hat{E} \Rightarrow \nabla \times \nabla \times \hat{A} = j\omega\epsilon\mu(-\nabla V - j\omega\hat{A}) \Rightarrow \nabla(\nabla \cdot \hat{A}) - \nabla^2\hat{A} = \omega^2\mu\epsilon\hat{A} - j\omega\epsilon\mu\nabla V$

$\Rightarrow \nabla^2\hat{A} + k^2\hat{A} = \nabla(\nabla\hat{A} + j\omega\mu\epsilon V) \Rightarrow$ let $\nabla \cdot \hat{A} = -j\omega\mu\epsilon V \Rightarrow V = j\frac{\nabla \cdot \hat{A}}{\omega\mu\epsilon} \Rightarrow \boxed{\hat{E} = -j\omega\hat{A} - j\frac{\nabla(\nabla \cdot \hat{A})}{\omega\mu\epsilon}}$

(d) $\hat{F} = \hat{a}_z\Psi_f(x,y,z)$ & $\hat{A} = 0 \Rightarrow \hat{F} = \hat{a}_z F_z(x,y,z)$ we have $\hat{H} = \hat{H}_A + \hat{H}_F = \frac{1}{\mu}\nabla \times \hat{A} - j\omega\hat{F} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla \cdot \hat{F})$

but $\hat{A} = 0 \Rightarrow \hat{H}_A = 0 \Rightarrow \hat{H} = -j\omega\hat{a}_z\Psi_f - \frac{j}{\omega\mu\epsilon}\nabla(\nabla \cdot \hat{a}_z\Psi_f) = -j\omega\hat{a}_z\Psi_f - \frac{j}{\omega\mu\epsilon}\underbrace{\nabla\left(\frac{\partial\Psi_f}{\partial z}\right)}_{\hat{a}_x\frac{\partial^2\Psi_f}{\partial x\partial z} + \hat{a}_y\frac{\partial^2\Psi_f}{\partial y\partial z} + \hat{a}_z\frac{\partial^2\Psi_f}{\partial z^2}}$

$\Rightarrow \boxed{H_x = \frac{-j}{\omega\mu\epsilon}\frac{\partial^2\Psi_f}{\partial x\partial z}}$ $\boxed{H_y = \frac{-j}{\omega\mu\epsilon}\frac{\partial^2\Psi_f}{\partial y\partial z}}$ $\boxed{H_z = -j\omega\Psi_f - \frac{j}{\omega\mu\epsilon}\frac{\partial^2\Psi_f}{\partial z^2} = \frac{-j}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\Psi_f}$

$\hat{E} = -\frac{1}{\epsilon}\nabla \times \hat{a}_z\Psi_f \Rightarrow$ $\boxed{E_z = 0}$ $\boxed{E_x = -\frac{1}{\epsilon}\frac{\partial\Psi_f}{\partial y}}$ $\boxed{E_y = \frac{1}{\epsilon}\frac{\partial\Psi_f}{\partial x}}$

Since $E_z = 0$, valid for TE & TEM waves . From the H_z expression $\frac{\partial^2}{\partial z^2} = -k_z^2$ & $\omega^2\mu\epsilon = k^2$

if $H_z = 0$

$H_z = 0$ iff $k^2 = k_z^2$ for TEM waves

(e) $\hat{A} = \hat{a}_z\Psi_a(x,y,z)$ & $\hat{F} = 0 \Rightarrow \hat{H} = \hat{H}_A$ and $\hat{E} = \hat{E}_A$

Similar to part (d) $\boxed{E_x = \frac{-j}{\omega\mu\epsilon}\frac{\partial^2\Psi_a}{\partial x\partial z}}$ $\boxed{E_y = \frac{-j}{\omega\mu\epsilon}\frac{\partial^2\Psi_a}{\partial y\partial z}}$ $\boxed{E_z = \frac{-j}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2} + \omega^2\mu\epsilon\right)\Psi_a}$

$\hat{H} = \frac{1}{\mu}\nabla \times \hat{A} \Rightarrow$ $\boxed{H_x = \frac{1}{\mu}\frac{\partial\Psi_a}{\partial y}}$ $\boxed{H_y = \frac{-1}{\mu}\frac{\partial\Psi_a}{\partial x}}$ $\boxed{H_z = 0}$

$H_z = 0$ valid for TM waves

if $E_z = 0 \Rightarrow$ TEM waves

$E_z = 0$ iff $\omega^2\mu\epsilon = -\frac{\partial^2}{\partial z^2} = k_z^2 \Rightarrow k = k_z$

(f) $\frac{E_x^+}{H_y^+} = \frac{-E_x^-}{H_y^-} = \sqrt{\frac{\mu}{\epsilon}}$ (as expected)