EEE 451 / 551

Solutions to HW # 8

1. The dominant mode is the $TE_{1,1}$ mode

(a) $(f_c)_{TE_{1,1}} = \dfrac{1.841}{2\pi a\sqrt{\mu\epsilon}} = 1.8315 \text{ GHz}.$

(b) $\lambda_g = \dfrac{\lambda}{\underbrace{\sqrt{1-(f_c/f)^2}}_{0.4017}}$ where $\lambda = \dfrac{\lambda_0}{\sqrt{\epsilon_r}} = 9.375 \text{ cm}$

$\Rightarrow \lambda_g = \dfrac{9.375}{0.4017} = 23.34 \text{ cm}$ ($\lambda_g > \lambda$ as expected)

(c) $k_z = k\sqrt{1-(f_c/f)^2} = 2\pi/\lambda \sqrt{1-(f_c/f)^2} = 26.92$ rad/m.

OR $k_z = \dfrac{2\pi}{\lambda_g} = \dfrac{2\pi}{\lambda_z} = 26.92$ rad/m

(d) $(Z_{TE})_{1,1} = \dfrac{\eta}{\sqrt{1-(f_c/f)^2}} = \dfrac{120\pi/\sqrt{2.56}}{0.4017} = 586.56\,\Omega$

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2. TE^z case, $F_z^+ = [A_1 J_n(\beta_\rho \rho) + B_1 Y_n(\beta_\rho \rho)][C_2 \cos(n\phi) + D_2 \sin(n\phi)]e^{-j\beta_z z}$ $\lim_{\rho\to 0} Y_n(\beta_\rho \rho) \to \infty \Rightarrow B_1 = 0$

$\bar{E} = -\frac{1}{\epsilon}\nabla\times\bar{F} \Rightarrow E_\rho = -\frac{1}{\epsilon}\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} \Rightarrow E_\rho = \frac{-A_1}{\epsilon}\frac{n}{\rho}J_n(\beta_\rho\rho)[-C_2\sin(n\phi) + D_2\cos(n\phi)]e^{-j\beta_z z}$

$E_\phi = \frac{1}{\epsilon}\frac{\partial F_z}{\partial \rho} \Rightarrow E_\phi = \frac{A_1}{\epsilon}\beta_\rho J_n'(\beta_\rho\rho)[C_2\cos(n\phi) + D_2\sin(n\phi)]e^{-j\beta_z z}$

B.C.s are $E_\rho(\rho, \phi=0, z) = 0$ —— (1) ; $E_\rho(\rho, \phi=\pi, z) = 0$ —— (2) ; $E_\phi(\rho=a, \phi, z) = 0$ —— (3)

From (1) $\underline{D_2 = 0}$, From (2) $\frac{-A_1}{\epsilon}\frac{n}{\rho}J_n(\beta_\rho\rho)[-C_2\sin(n\pi)]e^{-j\beta_z z} = 0 \Rightarrow$ <u>n should be an integer</u>

From (3) $\frac{A_{nm}}{\epsilon}\beta_\rho J_n'(\beta_\rho a)\cos(n\phi)e^{-j\beta_z z} = 0 \Rightarrow \beta_\rho a = p'_{nm}$ $m = 1,2,3$ $\Rightarrow \boxed{\beta_\rho = \frac{p'_{nm}}{a}}$

$\Rightarrow (f_c)_{n,m} = \dfrac{p'_{nm}}{2\pi a\sqrt{\mu\epsilon}}$

Then, the field expressions are:

from $\bar{H} = -j\omega\bar{F} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla\cdot\bar{F})$

$$\boxed{\begin{aligned} E_\rho &= A_{nm}\frac{1}{\epsilon}\frac{n}{\rho}J_n(\beta_\rho\rho)\sin(n\phi)e^{-j\beta_z z} \\ E_\phi &= A_{nm}\frac{\beta_\rho}{\epsilon}J_n'(\beta_\rho\rho)\cos(n\phi)e^{-j\beta_z z} \\ E_z &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} H_\rho &= \frac{-j}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial\rho\,\partial z} = -A_{nm}\frac{\beta_\rho\beta_z}{\omega\mu\epsilon}J_n'(\beta_\rho\rho)\cos(n\phi)e^{-j\beta_z z} \\ H_\phi &= \frac{-j}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial^2 F_z}{\partial\phi\,\partial z} = A_{nm}\frac{n\beta_z}{\omega\mu\epsilon}\frac{1}{\rho}J_n(\beta_\rho\rho)\sin(n\phi)e^{-j\beta_z z} \\ H_z &= \frac{-j}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2}+k^2\right)F_z = -j\frac{A_{nm}}{\omega\mu\epsilon}\beta_\rho^2 J_n(\beta_\rho\rho)\cos(n\phi)e^{-j\beta_z z} \end{aligned}}$$

3

(a) $\left(\nabla^2+k^2\right) F_y^+(x,y,z)=0 \Rightarrow F_y^+(x,y,z)=\left[C_1\cos(k_x x)+D_1\sin(k_x x)\right]\left[C_2\cos(k_y y)+D_2\sin(k_y y)\right] A_3 e^{-jk_z z}$

For each region, the solution for F_y^+ has the same form. Therefore,

$$F_y^d(x, 0\le y\le h, z)=\left[C_1^d\cos(k_{xd}x)+D_1^d\sin(k_{xd}x)\right]\left[C_2^d\cos(k_{yd}y)+D_2^d\sin(k_{yd}y)\right] A_3^d e^{-jk_z z}$$

with $k_{xd}^2+k_{yd}^2+k_z^2=k_d^2=\omega^2\mu_d\epsilon_d$

$$F_y^a(x, h\le y\le b, z)=\left[C_1^a\cos(k_{x0}x)+D_1^a\sin(k_{x0}x)\right]\left[C_2^a\cos(k_{y0}(b-y))+D_2^a\sin(k_{y0}(b-y))\right] A_3^a e^{-jk_z z}$$

with $k_{x0}^2+k_{y0}^2+k_z^2=k_0^2=\omega^2\mu_0\epsilon_0$

You should note that k_z is the same in both regions (the BCs will yield this result anyway)

(b) & (c) The boundary conditions are:

$E_z^d(x=0, 0\le y\le h, z)=E_z^d(x=a, 0\le y\le h, z)=0$ ——— B.C.1

$E_z^d(0\le x\le a, y=0, z)=0$ ——— B.C.2

$E_z^d(0\le x\le a, y=h, z)=E_z^a(0\le x\le a, y=h, z)$ ——— B.C.3

$E_z^a(x=0, h\le y\le b, z)=E_z^a(x=a, h\le y\le b, z)=0$ ——— B.C.4

$E_z^a(0\le x\le a, y=b, z)=0$ ——— B.C.5

$H_z^d(0\le x\le a, y=h, z)=H_z^a(0\le x\le a, y=h, z)$ ——— B.C.6

Note that using E_x^d, E_x^a, E_y^d, E_y^a and H_x^d & H_x^a you can obtain another set of boundary conditions. However, they will be dependent upon the above BCs. (B.C.1 – B.C.6).

The fields are obtained from $\bar{E}=-\frac{1}{\epsilon}\nabla\times\bar{F}$ and $\bar{H}=-j\omega\bar{F}-j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\bar{F})$ as

$E_x=\frac{1}{\epsilon}\frac{\partial F_y}{\partial z}$ $\qquad H_x=-j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_y}{\partial x\partial y}$

$E_y=0$ $\qquad H_y=-j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial y^2}+k^2\right)F_y$

$E_z=-\frac{1}{\epsilon}\frac{\partial F_y}{\partial x}$ $\qquad H_z=-j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_y}{\partial y\partial z}$

(d) (and (c))

From B.C. 3

$$\frac{\cancel{k_{x0}}}{\epsilon_0} A_{mn}^{a} \cancel{\sin(k_{x0}x)} \sin[k_{y0}(b-h)] \cancel{e^{-jk_z z}} = \frac{\cancel{\overset{k_{xd}}{\overset{\|}{k_{x0}}}}}{\epsilon_d} A_{mn}^{d} \cancel{\sin(\overset{k_{xd}}{\overset{\|}{k_{x0}}} x)} \sin(k_{yd}h) \cancel{e^{-jk_z z}}$$

$$\Rightarrow \boxed{\frac{1}{\epsilon_0} A_{mn}^{a} \sin[k_{y0}(b-h)] = \frac{1}{\epsilon_d} A_{mn}^{d} \sin(k_{yd}h)} \quad \text{——— (A)}$$

also

$$H_z^{a} = -j\frac{1}{\omega\mu_0\epsilon_0}\frac{\partial^2 F_y^{a}}{\partial y\,\partial z} = \frac{k_{y0}k_z}{\omega\mu_0\epsilon_0} A_{mn}^{a} \cos(k_{x0}x)\cos[k_{y0}(b-y)]\, e^{-jk_z z}$$

$$H_z^{d} = -j\frac{1}{\omega\mu_d\epsilon_d}\frac{\partial^2 F_y^{d}}{\partial y\,\partial z} = \frac{-k_{yd}k_z}{\omega\mu_d\epsilon_d} A_{mn}^{d} \cos(k_{xd}x)\cos(k_{yd}y)\, e^{-jk_z z}$$

From B.C. 6

$$\frac{k_{y0}\cancel{k_z}}{\cancel{\omega}\mu_0\epsilon_0} A_{mn}^{a} \cancel{\cos(k_{x0}x)} \cos[k_{y0}(b-h)] \cancel{e^{-jk_z z}} = \frac{-k_{yd}\cancel{k_z}}{\cancel{\omega}\mu_d\epsilon_d} A_{mn}^{d} \cancel{\cos(\underset{k_{x0}}{\underset{\|}{k_{xd}}}x)} \cos(k_{yd}h) \cancel{e^{-jk_z z}}$$

$$\Rightarrow \boxed{\frac{k_{y0}}{\mu_0\epsilon_0} A_{mn}^{a} \cos[k_{y0}(b-h)] = \frac{-k_{yd}}{\mu_d\epsilon_d} A_{mn}^{d} \cos(k_{yd}h)} \quad \text{——— (B)}$$

Dividing (B) to (A)

$$\frac{k_{y0}}{\mu_0}\cot[k_{y0}(b-h)] = \frac{-k_{yd}}{\mu_d}\cot[k_{yd}h]$$

and if $\mu_0 = \mu_d$

$$\Rightarrow \boxed{k_{yd}\tan[k_{y0}(b-h)] + k_{y0}\tan[k_{yd}h] = 0} \quad \text{——— (1)}$$

The second equation (as expected) coming from the propagation constants:

$$k_z^2 = \epsilon_r k_0^2 - k_{xd}^2 - k_{yd}^2 = k_0^2 - k_{x0}^2 - k_{y0}^2$$

where $k_{xd} = k_{x0} = \frac{m\pi}{a}$, $m = 0, 1, 2, 3, \ldots$

$$\Rightarrow \boxed{\epsilon_r k_0^2 - k_{yd}^2 = k_0^2 - k_{ya}^2} \quad \text{——— (2)}$$

(1) & (2) are the transcendental equations which should be solved simultaneously.

Then, $E_z^a = -\frac{1}{\epsilon_0}\frac{\partial F_y^a}{\partial x} = \frac{-k_{x0}}{\epsilon_0}\left[-C_1^a \sin(k_{x0}x) + D_1^a \cos(k_{x0}x)\right] \cdot$

$$\left\{C_2^a \cos[k_{y0}(b-y)] + D_2^a \sin[k_{y0}(b-y)]\right\} A_3^a e^{-jk_z z}$$

From B.C. 4 $\boxed{D_1^a = 0}$ and $\sin(k_{x0}a) = 0 \Rightarrow \boxed{k_{x0} = \frac{m\pi}{a}}$ $m = 0, 1, 2, \ldots$

From B.C. 5 $\boxed{C_2^a = 0}$

Then F_y^a reduces to $F_y^a = A_{mn}^a \cos(k_{x0}x) \sin[k_{y0}(b-y)] e^{-jk_z z}$

with $k_{x0} = \frac{m\pi}{a}$, $m = 0, 1, 2, 3, \ldots$

and $\left(\frac{m\pi}{a}\right)^2 + k_{y0}^2 + k_z^2 = k_0^2 = \omega^2 \mu_0 \epsilon_0$

and $E_z^a = \frac{k_{x0}}{\epsilon_0} A_{mn}^a \sin(k_{x0}x) \sin[k_{y0}(b-y)] e^{-jk_z z}$

For the dielectric region

$E_z^d = -\frac{1}{\epsilon_d}\frac{\partial F_y^d}{\partial x} = \frac{-k_{xd}}{\epsilon_d}\left[-C_1^d \sin(k_{xd}x) + D_1^d \cos(k_{xd}x)\right]\left[C_2^d \cos(k_{yd}y) + D_2^d \sin(k_{yd}y)\right]$

$\cdot A_3^d e^{-jk_z z}$

From B.C. 1 $\boxed{D_1^d = 0}$ and $\sin(k_{xd}a) = 0 \Rightarrow \boxed{k_{xd} = \frac{m\pi}{a}}$ $m = 0, 1, 2, \ldots$

From B.C. 2 $\boxed{C_2^d = 0}$

Then F_y^d reduces to $F_y^d = A_{mn}^d \cos(k_{xd}x) \sin(k_{yd}y) e^{-jk_z z}$

with $k_{xd} = \frac{m\pi}{a}$, $m = 0, 1, 2, 3, \ldots$

and $\left(\frac{m\pi}{a}\right)^2 + k_{yd}^2 + k_z^2 = k_d^2 = \omega^2 \mu_d \epsilon_d$

and $E_z^d = \frac{k_{xd}}{\epsilon_d} A_{mn}^d \sin(k_{xd}x) \sin(k_{yd}y) e^{-jk_z z}$

Also note that $k_{xd} = k_{x0} = \frac{m\pi}{a}$ as expected.