Solutions to HW#1

(i) $\nabla\times\bar{E} = -j\omega\mu\bar{H}$ $\qquad\qquad$ $\bar{B} = \nabla\times\bar{A}$

$\nabla\times\bar{H} = \bar{J}_i + \sigma\bar{E} + j\omega\epsilon\bar{E}$

$\nabla\cdot\bar{D} = \rho$

$\nabla\cdot\bar{B} = 0$

$$\nabla\times\bar{E} = -j\omega(\nabla\times\bar{A}) \Rightarrow \nabla\times\left[\bar{E}+j\omega\bar{A}\right]=0 \Rightarrow \bar{E}+j\omega\bar{A} = -\nabla V \Rightarrow \boxed{\bar{E} = -j\omega\bar{A}-\nabla V} \quad —(1)$$

$\nabla\times\bar{H} = \bar{J}_i + \sigma\bar{E} + j\omega\epsilon\bar{E}$ (use (1) in this equation)

$$\Rightarrow \nabla\times\bar{H} = \bar{J}_i + \sigma(-j\omega\bar{A}-\nabla V) + j\omega\epsilon(-j\omega\bar{A}-\nabla V)$$

$$\Rightarrow \nabla\times\bar{B} = \mu\bar{J}_i + \sigma\mu(-j\omega\bar{A}) - \sigma\mu\nabla V + \omega^2\mu\epsilon\bar{A} - j\omega\epsilon\mu\nabla V$$

$$\Rightarrow \nabla\times\nabla\times\bar{A} = \nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A} = \mu\bar{J}_i - j\sigma\mu\omega\bar{A} - \sigma\mu\nabla V + k^2\bar{A} - j\omega\epsilon\mu\nabla V$$

$$\Rightarrow \nabla\left[\nabla\cdot\bar{A} + \sigma\mu V + j\omega\epsilon\mu V\right] = \nabla^2\bar{A} + k^2\bar{A} - j\omega\sigma\mu\bar{A} + \mu\bar{J}_i$$

let $\nabla\cdot\bar{A} + \sigma\mu V + j\omega\epsilon\mu V = 0 \Rightarrow \boxed{\nabla\cdot\bar{A} = -\sigma\mu V - \epsilon\mu j\omega V}$ gauge

$$\Rightarrow \boxed{\nabla^2\bar{A} + k^2\bar{A} - j\omega\sigma\mu\bar{A} = -\mu\bar{J}_i}$$

(ii) $\nabla\cdot\bar{D} = \epsilon\nabla\cdot\bar{E} = \rho \Rightarrow \nabla\cdot\bar{E} = \rho/\epsilon \Rightarrow \nabla\cdot(-j\omega\bar{A}-\nabla V) = \rho/\epsilon$

$\Rightarrow \nabla^2 V + j\omega\nabla\cdot\bar{A} = -\rho/\epsilon$ (Substitute the obtained gauge)

$$\Rightarrow \nabla^2 V + j\omega(-\sigma\mu V - \epsilon\mu j\omega V) = -\rho/\epsilon \Rightarrow \boxed{\nabla^2 V + k^2 V - j\omega\mu\sigma V = -\rho/\epsilon}$$

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② $S = SWR = 1.9 \Rightarrow |\Gamma| = \dfrac{S-1}{S+1} = 0.31$

reflected power density $= |\Gamma|^2 P_{inc} \simeq 0.48\ W/m^2$

Transmitted " " $= (1-|\Gamma|^2)P_{inc} \simeq 4.52\ W/m^2$

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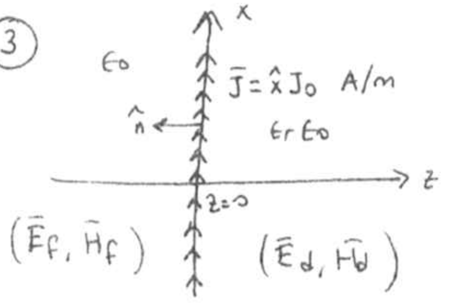
③



In dielectric medium

$$\bar{E}_d = \hat{a}_x E_0 e^{-jk_o\sqrt{\epsilon_r}z}$$

$$\bar{H}_d = \frac{1}{\eta_d}\hat{a}_k\times\bar{E}_d = \hat{a}_y\frac{E_0\sqrt{\epsilon_r}}{\eta_0}e^{-jk_o\sqrt{\epsilon_r}z}$$

with $\eta_d = \sqrt{\dfrac{\mu_0}{\epsilon_r\epsilon_0}}$ & $\eta_0 = \sqrt{\dfrac{\mu_0}{\epsilon_0}}$

In free-space medium

$$\bar{E}_f = \hat{a}_x E_0 e^{jk_o z} \Rightarrow \bar{H}_f = \frac{1}{\eta_0}\hat{a}_k\times\bar{E}_f = -\hat{a}_y\frac{E_0}{\eta_0}e^{jk_o z}$$

Apply BCs.

$$\bar{E}_d(z=0) = \bar{E}_f(z=0)$$

$$\hat{n}\times(\bar{H}_f - \bar{H}_d)\Big|_{z=0} = \bar{J} \Rightarrow -\hat{a}_z \times \left(-\hat{a}_y \frac{E_o}{\eta_o} - \hat{a}_y \frac{E_o\sqrt{\epsilon_r}}{\eta_o}\right) = \hat{a}_x J_o$$

$$\Rightarrow -\hat{a}_x \frac{E_o}{\eta_o}(1+\sqrt{\epsilon_r}) = \hat{a}_x J_o \Rightarrow \boxed{E_o = \frac{-\eta_o J_o}{1+\sqrt{\epsilon_r}}}$$

$\Rightarrow$ In dielectric medium
$$\bar{E}_d = -\hat{a}_x \frac{\eta_o J_o}{(1+\sqrt{\epsilon_r})} e^{-jk_o\sqrt{\epsilon_r}\,z}$$
$$\bar{H}_d = -\hat{a}_y \frac{J_o\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}} e^{-jk_o\sqrt{\epsilon_r}\,z}$$

$\Rightarrow$ In free-space medium
$$\bar{E}_f = -\hat{a}_x \frac{\eta_o J_o}{(1+\sqrt{\epsilon_r})} e^{jk_o z}$$
$$\bar{H}_f = \hat{a}_y \frac{J_o}{(1+\sqrt{\epsilon_r})} e^{jk_o z}$$

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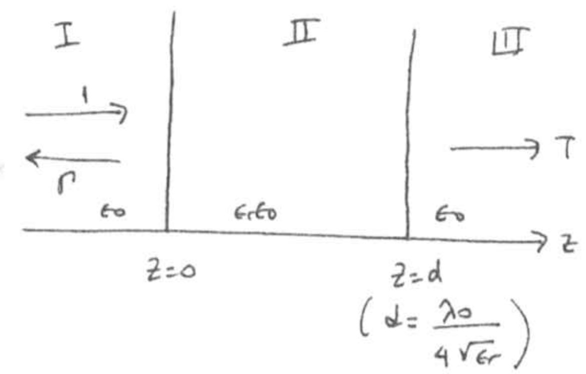

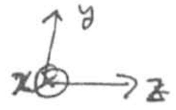

In Region 1
$$E_{1y} = e^{-jk_o z} + \Gamma e^{jk_o z}$$
$$H_{1x} = -\frac{1}{\eta_o} e^{-jk_o z} + \frac{\Gamma}{\eta_o} e^{jk_o z}$$

with $\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$, $\eta_1 = \sqrt{\frac{\mu_0}{\epsilon_r\epsilon_0}}$

$k_0 = \omega\sqrt{\mu_0\epsilon_0}$

$k_1 = k_0\sqrt{\epsilon_r} = \omega\sqrt{\mu_0\epsilon_r\epsilon_0}$

In Region 2
$$E_{2y} = A e^{-jk_1 z} + B e^{jk_1 z}$$
$$H_{2x} = -\frac{A}{\eta_1} e^{-jk_1 z} + \frac{B}{\eta_1} e^{jk_1 z}$$

In Region 3
$$E_{3y} = T e^{-jk_o(z-d)}$$
$$H_{3x} = \frac{-T}{\eta_o} e^{-jk_o(z-d)}$$

Apply BCs.

at $z=0$ $\quad E_{1y} = E_{2y} \Rightarrow \boxed{1+\Gamma = A+B}$ ——— (1)

$H_{1x} = H_{2x} \Rightarrow \boxed{\dfrac{\Gamma-1}{\eta_0} = \dfrac{B-A}{\eta_1}}$ ——— (2)

at $z=d$ $\quad E_{2y} = E_{3y} \Rightarrow A e^{-jk_1 d} + B e^{jk_1 d} = T \Rightarrow -jA + jB = T \quad \left(k_1 d = \frac{2\pi}{\lambda_1}\frac{\lambda_1}{4} = \right.$

$\Rightarrow \boxed{A-B = jT}$ ——— (3)

$H_{2x} = H_{3x} \Rightarrow \dfrac{-A}{\eta_1} e^{-jk_1 d} + \dfrac{B}{\eta_1} e^{jk_1 d} = \dfrac{-T}{\eta_0} \Rightarrow j\dfrac{A}{\eta_1} + \dfrac{jB}{\eta_1} = \dfrac{-T}{\eta_0}$

$\Rightarrow \boxed{A+B = jT\dfrac{\eta_1}{\eta_0}}$ ——— (4)

from (3) & (4)

$$\left.\begin{aligned} A-B &= jT \\ A+B &= j\frac{\eta_1}{\eta_0}T \end{aligned}\right\} \Rightarrow A = j\frac{T}{2}\left(\frac{\eta_0+\eta_1}{\eta_0}\right) \quad \& \quad B = j\frac{T}{2}\left(\frac{\eta_1-\eta_0}{\eta_0}\right)$$

Substitute A & B values into (1)

$$1+\Gamma = j\frac{T}{2}\left(\frac{\eta_0+\eta_1}{\eta_0} + \frac{\eta_1-\eta_0}{\eta_0}\right) \Rightarrow \boxed{1+\Gamma = jT\frac{\eta_1}{\eta_0}} \text{ ——— (5)}$$

Substitute A & B values into (2)

$$\Gamma-1 = \frac{\eta_0}{\eta_1}\left[\frac{jT}{2}\left(\frac{\eta_1-\eta_0}{\eta_0}\right) - j\frac{T}{2}\left(\frac{\eta_0+\eta_1}{\eta_0}\right)\right] \Rightarrow -1+\Gamma = \frac{\eta_0}{\eta_1}\frac{jT}{2}\left[\frac{\eta_1-\eta_0-\eta_0-\eta_1}{\eta_0}\right]$$

$$\Rightarrow \boxed{-1+\Gamma = \frac{-\eta_0}{\eta_1}jT} \text{ ——— (6)}$$

from (5) & (6)

$$\frac{1+\Gamma}{-1+\Gamma} = -\frac{\eta_1^2}{\eta_0^2} \Rightarrow \Gamma = \frac{-\eta_1^2/\eta_0^2+1}{-\eta_1^2/\eta_0^2-1} = \frac{-1/\epsilon_r+1}{-1/\epsilon_r-1} \Rightarrow \boxed{\Gamma = \frac{1-\epsilon_r}{1+\epsilon_r}}$$