Solutions to HW#2

① Rewrite Maxwell's equations as

$$\left(\nabla_t + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\bar{E}_t + \hat{z}E_z\right) = -j\omega\mu\left(\bar{H}_t + \hat{z}H_z\right), \quad \left(\nabla_t + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\bar{H}_t + \hat{z}H_z\right) = j\omega\epsilon\left(\bar{E}_t + \hat{z}E_z\right)$$

$\Rightarrow$ the transverse components:

$$\nabla_t \times \hat{z}E_z + \frac{\partial}{\partial z}\hat{z} \times \bar{E}_t = -j\omega\mu\bar{H}_t \quad \text{---} (1)$$

$$\nabla_t \times \hat{z}H_z + \frac{\partial}{\partial z}\hat{z} \times \bar{H}_t = j\omega\epsilon\bar{E}_t \quad \text{---} (2)$$

from (1) $\bar{H}_t = \frac{-1}{j\omega\mu}\left[\nabla_t \times \hat{z}E_z + \frac{\partial}{\partial z}\hat{z} \times \bar{E}_t\right]$ and substitute into (2)

$$\Rightarrow \nabla_t \times \hat{z}H_z + \frac{\partial}{\partial z}\hat{z} \times \left[\frac{-1}{j\omega\mu}\left(\nabla_t \times \hat{z}E_z + \frac{\partial}{\partial z}\hat{z} \times \bar{E}_t\right)\right] = j\omega\epsilon\bar{E}_t$$

$$\Rightarrow -j\omega\mu\nabla_t \times \hat{z}H_z + \frac{\partial}{\partial z}\hat{z} \times \left(\nabla_t \times \hat{z}E_z\right) + \underbrace{\frac{\partial^2}{\partial z^2}}_{-k_z^2}\hat{z} \times \left(\hat{z} \times \bar{E}_t\right) = \underbrace{\omega^2\mu\epsilon}_{k^2}\bar{E}_t$$

via the vector identity $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

$\hat{z} \times (\nabla_t \times \hat{z}) = \nabla_t$ and $\hat{z} \times (\hat{z} \times \bar{E}_t) = -\bar{E}_t$

$$\Rightarrow -j\omega\mu\nabla_t \times \hat{z}H_z + \frac{\partial}{\partial z}\nabla_t\bar{E}_z + k_z^2\bar{E}_t = k^2\bar{E}_t \Rightarrow \boxed{\bar{E}_t = \frac{1}{k^2-k_z^2}\left[\frac{\partial}{\partial z}\nabla_t E_z - j\omega\mu\nabla_t \times \hat{z}H_z\right]}$$

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② (a) $\bar{H}_t = \frac{1}{k^2-k_z^2}\left[\frac{\partial}{\partial z}\nabla_t H_z + j\omega\epsilon\nabla_t \times \hat{z}E_z\right]$ and $\bar{E}_t = \frac{1}{k^2-k_z^2}\left[\frac{\partial}{\partial z}\nabla_t E_z - j\omega\mu\nabla_t \times \hat{z}H_z\right]$

TM modes $\Rightarrow H_z = 0$

$$\Rightarrow \bar{H}_t = \frac{1}{k^2-k_z^2}\left[j\omega\epsilon\nabla_t \times \hat{z}E_z\right] \Rightarrow H_x = \frac{j\omega\epsilon}{k^2-k_z^2}\frac{\partial E_z}{\partial y} \quad \text{---} (1)$$

$$H_y = \frac{-j\omega\epsilon}{k^2-k_z^2}\frac{\partial E_z}{\partial x} \quad \text{---} (2)$$

$$\bar{E}_t = \frac{1}{k^2-k_z^2}\left[\frac{\partial}{\partial z}\nabla_t E_z\right] \Rightarrow E_x = \frac{-jk_z}{k^2-k_z^2}\frac{\partial E_z}{\partial x} \quad \text{---} (3)$$

$$E_y = \frac{-jk_z}{k^2-k_z^2}\frac{\partial E_z}{\partial y} \quad \text{---} (4)$$

$\frac{\partial}{\partial y} = 0$

$\Rightarrow$ (1) & (4) vanish

Then we have

$\boxed{E_z, E_x \;\&\; H_y}$ only

(b) In a lossless, source-free region, the system of 6 coupled scalar equations are:

$$\begin{aligned}
\cancel{\partial H_z/\partial y}^{\,0} - \partial H_y/\partial z &= \epsilon\,\partial E_x/\partial t \\
\cancel{\partial H_x/\partial z}^{\,0} - \cancel{\partial H_z/\partial x}^{\,0} &= \epsilon\,\cancel{\partial E_y/\partial t}^{\,0} \\
\partial H_y/\partial x - \cancel{\partial H_x/\partial y}^{\,0} &= \epsilon\,\partial E_z/\partial t \\
\cancel{\partial E_y/\partial z}^{\,0} - \cancel{\partial E_z/\partial y}^{\,0} &= \mu\,\cancel{\partial H_x/\partial t}^{\,0} \\
\partial E_z/\partial x - \partial E_x/\partial z &= \mu\,\partial H_y/\partial t \\
\cancel{\partial E_x/\partial y}^{\,0} - \cancel{\partial E_y/\partial x}^{\,0} &= \mu\,\cancel{\partial H_z/\partial t}^{\,0}
\end{aligned}$$

We have only E_z, E_x & H_y $\quad (E_y = H_x = H_z = 0)$

also $\frac{\partial}{\partial y} = 0$

$$\Rightarrow \boxed{\frac{\partial H_y}{\partial z} = -\epsilon\frac{\partial E_x}{\partial t}} \quad \text{---} (5)$$

$$\boxed{\frac{\partial H_y}{\partial x} = \epsilon\frac{\partial E_z}{\partial t}} \quad \text{---} (6)$$

$$\boxed{\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = \mu\frac{\partial H_y}{\partial t}} \quad \text{---} (7)$$

(c) $\frac{\partial}{\partial z} \leftrightarrow -jk_z \quad , \quad \frac{1}{c}\frac{\partial}{\partial t} \leftrightarrow jk \Rightarrow -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} = k^2$

$$\Rightarrow \nabla_t^2 H_y + (k^2 - k_z^2) H_y = 0 \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H_y + \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2}\right) H_y = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) H_y = 0$$

but $\frac{\partial}{\partial y} = 0$

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) H_y = 0}$$

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(3) $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \quad , \quad Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad \frac{R}{L} = \frac{G}{C}$ given

$$\Rightarrow \gamma = \sqrt{j\omega L\left(1 + \frac{R}{j\omega L}\right) j\omega C \left(1 + \frac{G}{j\omega C}\right)} = j\omega\sqrt{LC}\sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right) - \frac{RG}{\omega^2 LC}}$$

$$= j\omega\sqrt{LC}\sqrt{1 - 2j\frac{R}{\omega L} - \frac{R^2}{\omega^2 L^2}} = j\omega\sqrt{LC}\sqrt{\left(1 - j\frac{R}{\omega L}\right)^2} = j\omega\sqrt{LC}\left(1 - j\frac{R}{\omega L}\right)$$

$$\Rightarrow \gamma = R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC} = \alpha + j\beta \Rightarrow \boxed{\alpha = R\sqrt{\frac{C}{L}} \ , \ \beta = \omega\sqrt{LC}}$$

$$Z_0 = \sqrt{\frac{j\omega L\,(1 + R/j\omega L)}{j\omega C\,(1 + G/j\omega C)}} \qquad \frac{R}{L} = \frac{G}{C} \Rightarrow Z_0 = \sqrt{\frac{L}{C}} \Rightarrow \boxed{\begin{array}{l} R_0 = \sqrt{L/C} \\ X_0 = 0 \end{array}}$$

$\left.\begin{array}{ll} \beta = \omega\sqrt{LC} & \text{a linear function of frequency} \\ \alpha = R\sqrt{\frac{C}{L}} & \text{not a function of frequency} \end{array}\right\} \Rightarrow$ distortionless line

(b) $\left.\begin{array}{l} Z_0 = 50\,\Omega \Rightarrow R_0 = 50\,\Omega \ \& \ X_0 = 0 \\ \alpha = 0.01 \ (dB/m) \\ \beta = 0.8\pi \ (rad/m) \\ f = 100\ MHz \end{array}\right\}$ This is a distortionless line with $\alpha = R\sqrt{\frac{C}{L}}$, $\beta = \omega\sqrt{LC}$ and $Z_0 = \sqrt{\frac{L}{C}}$

$\alpha = 0.01\ dB/m \simeq 0.00115\ Np/m. \Rightarrow R = \alpha\sqrt{\frac{L}{C}} = \alpha Z_0 \simeq 0.00115 \times 50 = \underline{0.0575\,(\Omega/m)}$

$G = \frac{C}{L} R = \frac{\alpha^2}{R} = 2.3 \times 10^{-5} = \underline{23\,\mu S/m}$

$\beta Z_0 = \omega\sqrt{LC}\sqrt{\frac{L}{C}} \Rightarrow \beta Z_0 = \omega L \Rightarrow L = \frac{\beta Z_0}{\omega} = \underline{20\,nH/m}$

$Z_0 = \sqrt{\frac{L}{C}} \Rightarrow C = \frac{L}{Z_0^2} = \frac{20 \times 10^{-9}}{2500} = \underline{80\,pF/m}$