Chapter 2
Introduction to Differential Equations

2.1 Basic Definitions

An equation involving a real-valued function of one or more real independent variables and its derivatives (with respect to these variables) is called a **differential equation**. Some examples are

$$
\begin{aligned}
y' + (\ln x)y^2 &= 0\,, & y = y(x)\\
\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y &= \cos t\,, & y = y(t)\\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0\,, & u = u(x,y)\\
u_t - \alpha^2 u_{xx} &= 0\,, & u = u(t,x)
\end{aligned}
$$

A differential equation involving derivatives of a function of a single independent variable is called an **ordinary** differential equation, and one involving partial derivatives of a function of two or more independent variables is called a **partial** differential equation. First two equations above are ordinary differential equations, and the last two are partial differential equations. We will deal only with ordinary differential equations.

The **order** of a differential equation is the order of the highest derivative appearing in the equation. First equation above is a first order differential equation, the others are second order.

An nth order ordinary differential equation in a function y of an independent variable t is of the form

$$F(t, y, y', \ldots, y^{(n)}) = 0 \tag{2.1}$$

where F is a given real-valued function, and $y', y'', \ldots, y^{(n)}$ denote the first, second, and the nth derivative of y.[1] If $y^{(n)}$ can be written explicitly in terms of the remaining variables, then (2.1) becomes

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}) \tag{2.2}$$

[1] If the independent variable t appears explicitly in a differential equation, then it is understood that the dependent variable y is a function of t, and y', y'', etc. refer to the derivatives of y with respect to t. However, if the independent variable does not appear explicitly in the differential equation, then it is better to denote the derivatives of the dependent variable by $\frac{dy}{dt}, \frac{d^2y}{dt^2}$, etc., to indicate that the independent variable is t and y is a function of t.

A real-valued function $\phi(t)$ defined on an open interval $\mathcal{I} = (t_i, t_f)$ is called a **solution** of the differential equation in (2.1) if

$$F(t, \phi(t), \phi'(t), \ldots, \phi^{(n)}(t)) = 0 \quad \text{for all} \ \ t \in \mathcal{I}$$

Obviously, this requires that $\phi'(t), \ldots, \phi^{(n)}(t)$ and $F(t, \phi(t), \phi'(t), \ldots, \phi^{(n)}(t))$ exist for all $t \in \mathcal{I}$. The graph of $y = \phi(t)$ is called a **solution curve**.

Example 2.1

The function

$$\phi(t) = 1 + \sin t$$

is a solution of the differential equation

$$y'' + y = 1$$

on the interval $-\infty < t < \infty$, because $\phi'(t) = \cos t$ and $\phi''(t) = -\sin t$ exist and

$$\phi''(t) + \phi(t) = -\sin t + 1 + \sin t = 1$$

for all $-\infty < t < \infty$. The function

$$\psi(t) = 1 - 2\cos t$$

is also a solution of the same differential equation on the interval $-\infty < t < \infty$ as can be verified similarly.

Example 2.2

Any function of the form

$$\phi(t) = c/t$$

where c is a real number, is a solution of the differential equation

$$ty' + y = 0$$

on each of the intervals $-\infty < t < 0$ and $0 < t < \infty$. The solutions curves corresponding to different choices of c are shown in Figure 2.1.

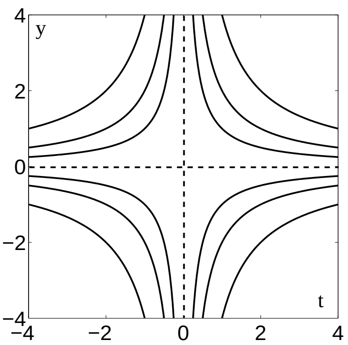

Figure 2.1: Solution curves of the differential equation in Example 2.2

If the differential equation in (2.1) can be written as

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = u(t) \tag{2.3}$$

where $a_1(t), \ldots, a_n(t)$ and $u(t)$ are given real-valued functions, then it is called a **linear** differential equation (LDE). If $u = 0$ in (2.3), then it is called **homogeneous**. We will deal mostly with linear differential equations having constant coefficients: $a_i(t) = a_i \in \mathbf{R}, i = 1, 2, \ldots, n$.

2.2 First Order LDE with Constant Coefficients

A first order linear differential equation with a constant coefficient is of the form

$$y' + ay = u(t)$$

We deal with the problem of solving the above differential equation in two steps: We first find a solution of a homogeneous equation, and then generate from it a solution of the non-homogeneous equation.

2.2.1 Homogeneous Equations

Consider the homogeneous equation

$$y' + ay = 0 \tag{2.4}$$

Clearly, $y = 0$ is a trivial solution of (2.4) for all t. In search of a nontrivial solution we rewrite the equation as $y' = -ay$, from which observe that the derivative of the solution must be a multiple of the solution itself. One such function is the exponential function. Motivated with this observation, we seek a solution of the form $y = e^{st}$ where s is a real number. Substituting y and $y' = se^{st}$ into (2.4) we get

$$se^{st} + ae^{st} = (s + a)e^{st} = 0$$

Since $e^{st} \neq 0$ for all t, we must have

$$s + a = 0$$

which is called the **characteristic equation** of (2.4). The characteristic equation has the root $s = -a$, which implies that $y = e^{-at}$ is a solution. But then it is easy to see that any multiple of e^{-at}, that is, any function of the form

$$y = ce^{-at}, \quad c \in \mathbf{R} \tag{2.5}$$

is also a solution for all t. The constant c in expression (2.5) can be chosen arbitrarily, and for each choice of c we get a different solution as shown in Figure 2.2 for $a = 1$. Thus the expression in (2.5) defines a one-parameter family of solutions.

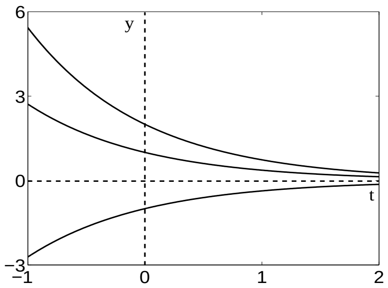

Figure 2.2: Solutions of (2.4) for $a = 1$.

2.2.2 Non-homogeneous Equations

Now consider the non-homogeneous equation

$$y' + ay = u(t) \tag{2.6}$$

We replace the constant c in (2.5) with a function $v(t)$, and look for a solution of the form $y = e^{-at}v(t)$.[2] Substituting y and $y' = e^{-at}v'(t) - ae^{-at}v(t)$ into (2.6), we get

$$e^{-at}v'(t) - ae^{-at}v(t) + ae^{-at}v(t) = e^{-at}v'(t) = u(t)$$

or equivalently,

$$v'(t) = e^{at}u(t)$$

Thus

$$v(t) = \int e^{at}u(t)dt = V(t) + c$$

where $V(t)$ is any antiderivative of $v'(t) = e^{at}u(t)$, and $c \in \mathbf{R}$ is an arbitrary constant. Hence any function of the form

$$y = e^{-at}[V(t) + c]\,, \quad c \in \mathbf{R} \tag{2.7}$$

is a solution of the non-homogeneous equation in (2.6).[3]

As in the homogeneous case, the expression in (2.7) contains an arbitrary constant c, and thus defines a family of solutions. By analogy to the solution of linear systems considered in Chapter 1, any member of this family obtained by assigning a fixed value to the arbitrary constant c is called a **particular solution**, denoted ϕ_p. A simple particular solution is obtained by choosing $c = 0$ as

$$\phi_p(t) = e^{-at}V(t)$$

[2] This method of finding a solution of a non-homogeneous linear differential equation from a solution of the associated homogeneous equation is known as the method of **variation of parameters**, and is also applicable to higher order differential equations.

[3] From now on, we will omit the phrase "$c \in \mathbf{R}$" from a solution expression for simplicity in notation.

Then the family of solutions in (2.7) can be written as

$$y = e^{-at}V(t) + ce^{-at} = \phi_p(t) + \phi_c(t) \tag{2.8}$$

where

$$\phi_c(t) = ce^{-at}$$

characterizes a family of solutions of the homogeneous equation (2.4) associated with (2.6). ϕ_c is called a **complementary solution** of (2.6) because by adding to ϕ_p any member of the family defined by ϕ_c we obtain another particular solution.

Example 2.3

A complementary solution of the differential equation

$$y' + 2y = 5\cos t \tag{2.9}$$

is

$$\phi_c(t) = ce^{-2t}$$

To find a particular solution, we substitute $y = e^{-2t}v(t)$ and its derivative into the given equation, and obtain

$$e^{-2t}v'(t) - 2e^{-2t}v(t) + 2e^{-2t}v(t) = 5\cos t$$

or equivalently,

$$v'(t) = 5e^{2t}\cos t$$

Taking the antiderivative of both sides, we get

$$v(t) = \int 5e^{2t}\cos t\,dt = e^{2t}(2\cos t + \sin t) + c$$

Thus a family of solutions is obtained as

$$y = e^{-2t}v(t) = 2\cos t + \sin t + ce^{-2t}$$

where

$$\phi_p(t) = 2\cos t + \sin t \tag{2.10}$$

is a particular solution.

Now consider the same differential equation with a different right-hand side:

$$y' + 2y = 4t \tag{2.11}$$

Complementary solution is still $\phi_c(t) = ce^{-2t}$. Following the same steps as above, a family of solutions can be found as

$$y = 2t - 1 + ce^{-2t}$$

where

$$\phi_p(t) = 2t - 1 \tag{2.12}$$

is a particular solution.

What if the right-hand side of the differential equation is the sum of the right-hand sides of (2.9) and (2.11)? The reader might suspect that a particular solution would be the sum of the particular solutions in (2.10) and (2.12). Indeed, it is easy to verify that

$$y = 2\cos t + \sin t + 2t - 1$$

is a particular solution of

$$y' + 2y = 5\cos t + 4t$$

So far we have shown that any member of the family defined by (2.8) is a solution of the differential equation (2.6), but left the question whether there may be other solutions that do not belong to this family unanswered. We will consider this issue in the next section.

2.3 Initial Conditions

Suppose that we are interested in finding among the family of solutions given in (2.8) a particular one which has the value $y = y_0$ when $t = t_0$. In other words, we look for a particular solution whose graph passes through the point (t_0, y_0) in the ty plane. Such a condition is called an **initial condition**, and a differential equation with an initial condition attached to it is called an **initial-value problem**. We describe an initial-value problem involving a first order differential equation as

$$y' + ay = u(t)\,, \quad y(t_0) = y_0 \tag{2.13}$$

A function $\phi(t)$ is called a solution of the initial-value problem in (2.13) on an interval $\mathcal{I}$ that includes t_0 if it is a solution of the differential equation on $\mathcal{I}$ and $\phi(t_0) = y_0$.

If we assume that the solution of the initial-value problem in (2.13) is included in the family of solutions given by (2.8), then to find it all we have to do is to fix the arbitrary constant c in expression (2.8) to satisfy the initial condition as we illustrate by the following example.

Example 2.4

Let us solve the initial-value problem

$$y' + 2y = 5\cos t\,, \quad y(0) = 1$$

A family of solutions of the differential equation has already been obtained in Example 2.3 as

$$y = 2\cos t + \sin t + ce^{-2t}$$

To evaluate the arbitrary constant, we substitute $t_0 = 0$ for t and $y_0 = 1$ for y, and get

$$1 = 2\cos 0 + \sin 0 + ce^{-2.0} = 2 + c$$

which gives $c = -1$. Thus the solution of the initial-value problem is obtained as

$$y = 2\cos t + \sin t - e^{-2t}$$

There are two questions concerning the initial-value problem in (2.13).

a) Under what conditions does there exist a solution?

b) If a solution exists, is it included in the family of solutions given by (2.8)?

The first question is answered by the following theorem, whose proof is given in Appendix B for a more general case.

Theorem 2.1 *Suppose that the function $u(t)$ is piece-wise continuous on an interval $\mathcal{I}$ which includes t_0.*[4] *Then there exists a unique continuous function $\phi(t)$ defined on $\mathcal{I}$ such that $\phi(t_0) = y_0$ and $y = \phi(t)$ is a solution of the differential equation in (2.13) on every subinterval of $\mathcal{I}$ that does not contain a discontinuity point of $u(t)$.*

The function $\phi(t)$ in the statement of Theorem 2.1 satisfies the differential equation (2.13) for all $t \in \mathcal{I}$ except the discontinuity points of $u(t)$, where $\phi(t)$ is well defined (as it is continuous) but $\phi'(t)$ fails to exist. However, since there are only a finite number of such points in every finite subinterval of $\mathcal{I}$, we can extend the definition of solution to include such piece-wise differentiable functions. With this extended definition of solution, Theorem 2.1 states that the initial-value problem in (2.13) has a unique continuous, piece-wise differentiable solution $y = \phi(t)$ on $\mathcal{I}$.

Note that the theorem tells more than the existence of a solution. It also states that the solution is unique. It is the uniqueness of the solution that allows us to answer the second question.

Consider the function

$$\phi(t) = e^{-at}[V(t) - V(t_0) + e^{at_0}y_0] \tag{2.14}$$

which is a particular solution of the differential equation (2.13) obtained from (2.8) by choosing $c = e^{at_0}y_0 - V(t_0)$. Evaluating this function at $t = t_0$, we get

$$\phi(t_0) = e^{-at_0}[V(t_0) - V(t_0) + e^{at_0}y_0] = y_0$$

that is, ϕ also satisfies the initial condition. Then it must be the unique solution of (2.13). This shows that the solution of the initial-value problem (2.13) is indeed included in the family of solutions given by (2.8).

The reader may wonder how ϕ can be unique while V can be chosen to be any antiderivative of v'. The answer is that although $V(t)$ is not unique, $V(t) - V(t_0)$ is, because all antiderivatives differ only by a constant. If $\tilde{V}$ is any other antiderivative, then $\tilde{V}(t) = V(t) + C$ and $\tilde{V}(t_0) = V(t_0) + C$, so that $\tilde{V}(t) - \tilde{V}(t_0) = V(t) - V(t_0)$. A convenient choice for V is given by the definite integral

$$V(t) = \int_{t_0}^{t} e^{a\tau}u(\tau)d\tau$$

for which $V(t_0) = 0$. With this choice of $V(t)$, the unique solution of (2.13) is obtained from (2.14) as

$$\phi(t) = e^{-a(t-t_0)}y_0 + e^{-at}\int_{t_0}^{t} e^{a\tau}u(\tau)\,d\tau = \phi_o(t) + \phi_u(t) \tag{2.15}$$

[4] A function $f(t)$ defined on a finite interval is said to be piece-wise continuous if it is continuous everywhere except for a *finite* number of discontinuity points, and left and right limits of f exist at the discontinuity points. A function defined on an infinite or semi-infinite interval is piece-wise continuous if it is piece-wise continuous on every finite subinterval.

This expression gives the solution as the sum of two parts; one part, $\phi_o(t)$, due to the initial condition y_0, and the other part, $\phi_u(t)$, due to the forcing function $u(t)$.[5]

Let us go back to the non-homogeneous differential equation (2.6). Let $y = \psi(t)$ be a solution of (2.6) on an interval $\mathcal{I}$, and let $\psi(t_0) = \psi_0$ at some arbitrary $t_0 \in \mathcal{I}$. Then obviously $y = \psi(t)$ is a solution of the initial-value problem

$$y' + ay = u(t)\,, \quad y(t_0) = \psi_0$$

and, by the above discussion, it must be included in the family of solutions given by (2.8). This shows that the expression (2.8) includes all possible solutions of (2.6). Because of this reason it is called a **general solution** of (2.6). Note that since $V(t)$ in (2.8) is not unique, a general solution may be expressed in many different ways.

Example 2.5

Let us find the solution of the initial-value problem

$$y' + ay = au(t)\,, \quad y(t_0) = y_0 \tag{2.16}$$

where $a \neq 0$, and

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

Such a function u is called a **unit step** function, and is common in many engineering applications. Note that unit step function is continuous everywhere except $t = 0$, where it has a jump as shown in Figure 2.3.

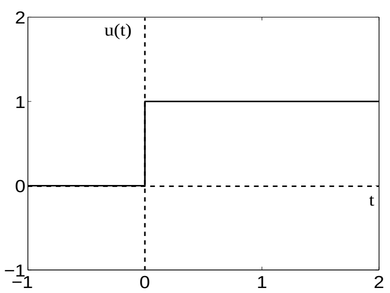

Figure 2.3: Unit step function

Since $u(t)$ has a discontinuity at $t = 0$, it is reasonable to look for separate solutions on the intervals $-\infty < t < 0$ and $0 < t < \infty$.

On the interval $-\infty < t < 0$, $u(t) = 0$, and a general solution of the resulting homogeneous differential equation

$$y' + ay = 0$$

is given as

$$y = c_1 e^{-at}\,, \quad t < 0 \tag{2.17}$$

[5]Unlike the decomposition of a solution into particular and complementary solutions as in (2.8), the parts $\phi_o(t)$ and $\phi_u(t)$ in (2.15) are not themselves solutions of (2.13).

On the interval $0 < t < \infty$, $u(t) = 1$, and the differential equation becomes

$$y' + ay = a$$

Now a general solution is

$$y = 1 + c_2 e^{-at}, \quad t > 0 \tag{2.18}$$

The solution curves defined by (2.17) and (2.18) are shown in Figure 2.4 for $a = 1$. Note that these solutions are not defined at $t = 0$. However, for any given y_0, there exist a particular solution ϕ_1 in the family defined by (2.17) and a particular solution ϕ_2 in the family defined by (2.18) such that

$$\lim_{t \to 0^-} \phi_1(t) = y_0 = \lim_{t \to 0^+} \phi_2(t) \tag{2.19}$$

The first condition in (2.19) requires that $c_1 = y_0$. Hence

$$\phi_1(t) = e^{-at} y_0, \quad t < 0$$

Similarly, using the second condition in (2.19) to evaluate c_2 in (2.18), we get

$$\phi_2(t) = e^{-at} y_0 + 1 - e^{-at}, \quad t > 0$$

Combining ϕ_1 and ϕ_2 and extending their domains of definition to include $t = 0$, we obtain a single solution for all $-\infty < t < \infty$ as

$$y = \begin{cases} e^{-at} y_0, & t \leq 0 \\ e^{-at} y_0 + 1 - e^{-at}, & t \geq 0 \end{cases} \tag{2.20}$$

Note that the solution is continuous at all t, and it satisfies the given differential equation at all t except $t = 0$. This is what we mean by a solution in the extended sense of Theorem 2.1.

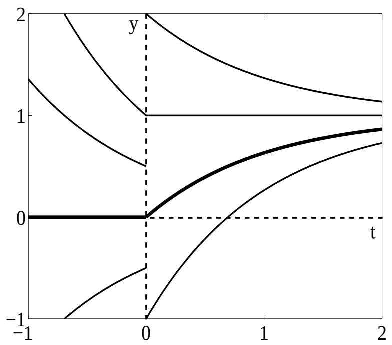

Figure 2.4: Solution curves of (2.16) for $a = 1$

The separate treatment of the cases $t < 0$ and $t > 0$ can be avoided by using the solution expression given in (2.15). If $t < 0$ then $u(\tau) = 0$ for all $t < \tau < 0$, and therefore, the integral term is zero. The solution is then given by

$$y = e^{-at} y_0, \quad t \leq 0$$

If $t > 0$, $u(\tau) = 1$ for all $0 < \tau < t$, and the solution is found as

$$y = e^{-at}y_0 + e^{-at}\int_0^t ae^{a\tau}d\tau = e^{-at}y_0 + 1 - e^{-at}\,, \quad t \geq 0$$

Of particular interest is the case when $y_0 = 0$. In this case, the solution becomes

$$y = \begin{cases} 0, & t \leq 0 \\ 1 - e^{-at}, & t \geq 0 \end{cases}$$

which is called the **step response** of the differential equation, and is indicated by the thick curve in Figure 2.4.

Example 2.6

Let us consider the initial-value problem

$$y' + y = u_T(t)\,, \quad y(0) = 0 \tag{2.21}$$

where

$$u_T(t) = \begin{cases} 0, & t < 0 \quad \text{or} \quad t > T \\ 1/T, & 0 < t < T \end{cases} \tag{2.22}$$

Note that the area under the graph of $u_T(t)$ is 1 as shown in Figure 2.5. Such a function is called a **unit pulse**.

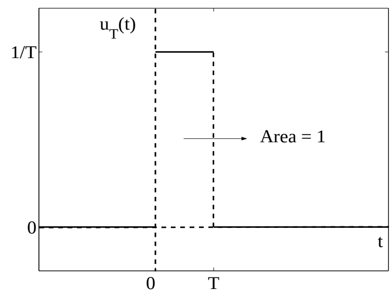

Figure 2.5: Unit pulse function.

As in the previous example, $y = 0$ for $t \leq 0$.
If $0 < t \leq T$, then $u_T(\tau) = 1/T$ for all $0 < \tau < t$, and

$$y = e^{-t}\int_0^t (1/T)e^{\tau}d\tau = \frac{1 - e^{-t}}{T}$$

If $t \geq T$, $u_T(\tau)$ contributes to the integral only for $0 < \tau < T$, so that

$$y = e^{-t}\int_0^T (1/T)e^{\tau}d\tau = \frac{e^T - 1}{T}\,e^{-t}$$

The solution is shown in Figure 2.6 for several values of T.

It is interesting to examine the behavior of $u_T(t)$ and the solution as $T \to 0$. As T gets smaller, the height of the pulse $u_T(t)$ tends to ∞ at $t = 0$ and to 0 everywhere else, while the area under the pulse remains unchanged. The limit of u_T is called a **unit impulse**, denoted $\delta(t)$.[6] Now the corresponding solution tends to[7]

$$y = \frac{e^T - 1}{T}\, e^{-t} \quad \rightarrow \quad e^{-t}\,, \quad t > 0$$

We formally say that the initial-value problem

$$y' + y = \delta(t)\,, \quad y(0^-) = 0$$

has the solution

$$y = \begin{cases} 0, & t < 0 \\ e^{-t}, & t > 0 \end{cases}$$

which we call the **impulse response** of the differential equation. Note that the impulse response is not continuous, but has a jump at $t = 0$, as indicated by the thick solution curve in Figure 2.6. Since impulse is not a piece-wise continuous function, we should not expect to get a continuous solution. Theorem 2.1 is not applicable to this case. Note also that the initial condition is specified not exactly at $t = 0$, but at some $t = -\epsilon$ where $\epsilon > 0$ is arbitrarily small (which is denoted by 0^- for convenience). The reason is that we do not know what is going on at $t = 0$. Looking at the solution for $t > 0$, we observe that it is actually the same as the solution of the homogeneous initial-value problem

$$y' + y = 0\,, \quad y(0) = 1$$

It looks as if the impulse has changed the initial condition from 0 to 1 instantaneously at $t = 0$.

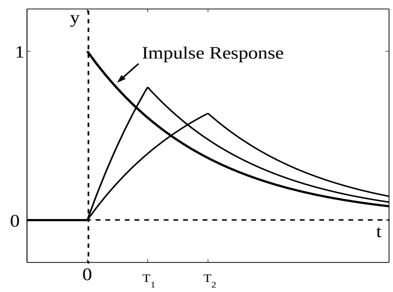

Figure 2.6: Solution of (2.21) for several T.

[6]Unit impulse is not a function in the ordinary sense, because it is not defined at $t = 0$, it is zero everywhere except $t = 0$, and yet

$$\int_{-\epsilon}^{\epsilon} \delta(t)dt = 1$$

for any $\epsilon > 0$.

[7]This follows from

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Example 2.7

In the previous two examples the point at which the initial condition was specified was also a discontinuity point of the forcing function $u(t)$, but these were just coincidence and irrelevant for the solution formula in (2.15). To illustrate this point consider the initial-value problem

$$y' + y = u(t)\,, \quad y(0) = y_0 \tag{2.23}$$

where

$$u(t) = \begin{cases} 1, & t > 1 \\ 0, & t < 1 \end{cases}$$

is a shifted unit step function.

For $t < 1$

$$y \;=\; \phi_1(t) \;=\; e^{-t}y_0 + e^{-t}\int_0^t e^{\tau}u(\tau)\,d\tau \;=\; e^{-t}y_0$$

where the second equality follows from the fact that $u(\tau) = 0$ on the interval of integration. For $t > 1$

$$\begin{aligned} y \;=\; \phi_2(t) \;&=\; e^{-t}y_0 + e^{-t}\int_0^t e^{\tau}u(\tau)\,d\tau \\ &=\; e^{-t}y_0 + e^{-t}\int_1^t e^{\tau}\,d\tau \;=\; e^{-t}y_0 + 1 - e^{-(t-1)} \end{aligned}$$

Note that

$$\lim_{t \to 1^-} \phi_1(t) = y_0/e = \lim_{t \to 1^+} \phi_2(t)$$

Combining ϕ_1 and ϕ_2 after extending their domains to include the discontinuity point $t = 1$ of u, we obtain a continuous solution whose graph is shown in Figure 2.7 for $y_0 = 0.4$.

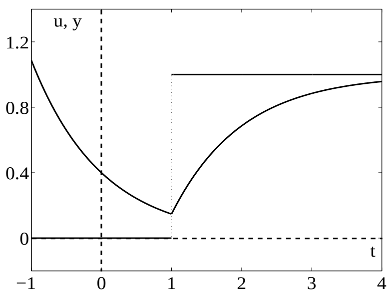

Figure 2.7: Solution of (2.23)

The formula in (2.15) provides us with a nice property of the solution of an initial-value problem involving a first order linear differential equation with a constant coefficient: If the initial-value problem

$$y' + ay = u(t)\,, \quad y(0) = y_0$$

has the solution $y = \phi(t)$, then the initial-value problem

$$y' + ay = u(t - t_0)\,, \quad y(t_0) = y_0$$

has the solution $y = \psi(t) = \phi(t - t_0)$ (see Exercise 2.6). This property allows us to derive the solution of a differential equation with an initial condition specified at some arbitrary t_0 from the solution of a modified differential equation with an initial condition specified at $t_0 = 0$. (Modification involves an appropriate shift of the forcing function $u(t)$.) It should however be emphasized that this property does not hold for a differential equation with a non-constant coefficient.

Example 2.8

The initial-value problem

$$y' + y = 2\,\cos t\,, \quad y(0) = 1$$

has the solution

$$y = \phi(t) = \sin t + \cos t$$

as can be verified by observing that

$$\phi'(t) + \phi(t) = (\cos t - \sin t) + (\sin t + \cos t) = 2\,\cos t$$

and

$$\phi(0) = \sin 0 + \cos 0 = 1$$

Then the initial-value problem

$$y' + y = 2\,\cos(t - \pi/2) = 2\,\sin t\,, \quad y(\pi/2) = 1$$

must have the solution

$$y = \psi(t) = \phi(t - \pi/2) = \sin(t - \pi/2) + \cos(t - \pi/2) = \sin t - \cos t$$

Indeed,

$$\psi'(t) + \psi(t) = (\cos t + \sin t) + (\sin t - \cos t) = 2\,\sin t$$

and

$$\psi(\pi/2) = \sin \pi/2 - \cos \pi/2 = 1$$

2.4 Second Order LDE with Constant Coefficients

2.4.1 Homogeneous Second Order Equations

Again we start with the homogeneous equation

$$y'' + a_1 y' + a_2 y = 0 \tag{2.24}$$

where $a_1, a_2 \in \mathbf{R}$ are given constants. As in the first order equations, $y = 0$ is a trivial solution, and we look for nontrivial solutions of the form $y = e^{st}$. Substituting y and its derivatives into the equation, we get the characteristic equation

$$s^2 + a_1 s + a_2 = 0$$

The characteristic equation is a second order equation with real coefficients. According to the fundamental theorem of algebra, it has two roots which may be real or complex. We investigate three possible cases separately.

Characteristic equation has distinct real roots

If $a_1^2 - 4a_2 > 0$, then the characteristic equation has two distinct real roots, $s = \sigma_1$ and $s = \sigma_2$. Then each of the functions

$$\phi_1(t) = e^{\sigma_1 t} \quad \text{and} \quad \phi_2(t) = e^{\sigma_2 t}$$

is a solution of the differential equation (2.24). Moreover, any function of the form

$$y = c_1\phi_1(t) + c_2\phi_2(t) = c_1 e^{\sigma_1 t} + c_2 e^{\sigma_2 t}$$

where $c_1, c_2 \in \mathbf{R}$ are arbitrary constants, is also a solution as can easily be verified by substitution.

The solutions ϕ_1 and ϕ_2 have the property that neither of them can be expressed as a multiple of the other, and are said to be **linearly independent**. The importance of linear independence of ϕ_1 and ϕ_2 is that the solution expression above cannot be simplified by combining the two terms, which means that the two arbitrary constants c_1 and c_2 can be assigned arbitrary values independently, and we get a different solution for every different choice of the pair (c_1, c_2).[8]

Characteristic equation has complex roots

If $a_1^2 - 4a_2 < 0$, then the characteristic equation has a pair of complex conjugate roots $s = \lambda_1 = \sigma + i\omega$ and $s = \lambda_1^* = \sigma - i\omega$, where $\sigma, \omega \in \mathbf{R}$. Then each of the complex-valued functions

$$\psi_1(t) = e^{(\sigma+i\omega)t} = e^{\sigma t}(\cos\omega t + i\sin\omega t)$$

and

$$\psi_2(t) = \psi_1^*(t) = e^{(\sigma-i\omega)t} = e^{\sigma t}(\cos\omega t - i\sin\omega t)$$

satisfies the differential equation (2.24), and is called a **complex solution**. To find real solutions we write $\psi_1(t) = \phi_1(t) + i\phi_2(t)$ where the real and imaginary parts

$$\phi_1(t) = e^{\sigma t}\cos\omega t \quad \text{and} \quad \phi_2(t) = e^{\sigma t}\sin\omega t$$

of ψ_1 are real-valued functions. Since ψ_1 satisfies the differential equation, we have

$$0 = \psi_1'' + a_1\psi_1' + a_2\psi_1 = (\phi_1'' + a_1\phi_1' + a_2\phi_1) + i(\phi_2'' + a_1\phi_2' + a_2\phi_2)$$

which implies

$$\phi_1'' + a_1\phi_1' + a_2\phi_1 = 0$$

and

$$\phi_2'' + a_1\phi_2' + a_2\phi_2 = 0$$

where the argument t of the functions are dropped for convenience. This shows that both the real part ϕ_1 and the imaginary part ϕ_2 of ψ_1 are solutions of (2.24).

[8]Recall that we talked about linear independence of column vectors in Chapter 1 in connection with solution of linear systems. Here the same concept is used for functions. A precise definition of linear independence of functions will be given in Chapter 3.

The reader can also verify this by substituting ϕ_1 and its derivatives and ϕ_2 and its derivatives into (2.24) (see Exercise 2.8). We would reach the same result if we considered ψ_2 instead of ψ_1, because their real parts are the same and imaginary parts differ only in sign.

The functions $\phi_1(t) = e^{\sigma t}\cos\omega t$ and $\phi_2(t) = e^{\sigma t}\sin\omega t$ are linearly independent, and any function of the form

$$y = c_1\phi_1(t) + c_2\phi_2(t) = c_1 e^{\sigma t}\cos\omega t + c_2 e^{\sigma t}\sin\omega t$$

is also a solution.

Characteristic equation has a double real root

If $a_1^2 - 4a_2 = 0$, then the characteristic equation is of the form

$$s^2 - 2\sigma s + \sigma^2 = (s-\sigma)^2 = 0$$

and it has a double root at $s = \sigma$. In this case the function $\phi_1(t) = e^{\sigma t}$ is a solution of (2.24). Since there is no reason to think that this case is any different from the previous two cases in an essential way, we look for a second solution which is linearly independent of ϕ_1. To find a second solution, we define a new dependent variable as $x = y' - \sigma y$. Then the differential equation becomes

$$y'' - 2\sigma y' + \sigma^2 y = (y'' - \sigma y') - \sigma(y' - \sigma y) = x' - \sigma x = 0$$

Thus the original second order equation in y is reduced to a first order equation in the new variable x. The solution of this equation is

$$x = c_2 e^{\sigma t}$$

Substituting this expression in the equation defining x in terms of y, we obtain

$$y' - \sigma y = c_2 e^{\sigma t}$$

which is another first order differential equation in y, but now it is a non-homogeneous one. Solving this equation we obtain

$$y = c_1 e^{\sigma t} + c_2 t e^{\sigma t}$$

This expression is already in the form $y = c_1\phi_1(t) + c_2\phi_2(t)$, where $\phi_1(t) = e^{\sigma t}$ and $\phi_2(t) = te^{\sigma t}$. Thus we not only recover ϕ_1, but also obtain a second solution ϕ_2. As in the previous two cases, ϕ_1 and ϕ_2 are linearly independent.

In summary, the second order homogeneous linear differential equation (2.24) has a solution of the form

$$y = c_1\phi_1(t) + c_2\phi_2(t) \tag{2.25}$$

where ϕ_1 and ϕ_2 are linearly independent solutions that are defined by the roots of the characteristic equation.

2.4.2 Non-homogeneous Second Order Equations

We now consider the non-homogeneous equation

$$y'' + a_1 y' + a_2 y = u(t) \tag{2.26}$$

As in the first order equation, we use the method of variation of parameters, and assume a solution of the form

$$y = \phi_1(t)v_1(t) + \phi_2(t)v_2(t)$$

where ϕ_1 and ϕ_2 are linearly independent solutions of the associated homogeneous equation, and the functions v_1 and v_2 are to be determined. The derivative of y is obtained as

$$y' = \phi_1' v_1 + \phi_1 v_1' + \phi_2' v_2 + \phi_2 v_2'$$

where the argument t is dropped for simplicity. Let us impose the condition

$$\phi_1(t)v_1'(t) + \phi_2(t)v_2'(t) = 0 \tag{2.27}$$

on v_1 and v_2.[9] Then the expression for y' reduces to

$$y' = \phi_1' v_1 + \phi_2' v_2$$

and differentiating once more we get

$$y'' = \phi_1'' v_1 + \phi_1' v_1' + \phi_2'' v_2 + \phi_2' v_2'$$

Substituting y, y', and y'' into the differential equation in (2.26) and grouping the terms, we obtain

$$\begin{aligned} & y'' + a_1 y' + a_2 y \\ & \quad = \quad (\phi_1'' + a_1\phi_1' + a_2\phi_1)v_1 + (\phi_2'' + a_1\phi_2' + a_2\phi_2)v_2 + \phi_1' v_1' + \phi_2' v_2' \\ & \quad = \quad \phi_1' v_1' + \phi_2' v_2' \quad = \quad u \end{aligned}$$

where the second equation follows from the fact that ϕ_1 and ϕ_2 are solutions of the homogeneous part of (2.26) so that the expressions in the parentheses are zero. Thus we obtain a second equation in v_1' and v_2'

$$\phi_1'(t)v_1'(t) + \phi_2'(t)v_2'(t) = u(t) \tag{2.28}$$

Linear independence of ϕ_1 and ϕ_2 guarantee that equations (2.27) and (2.28) can be solved simultaneously for v_1' and v_2'.[10] The reader can verify that

$$v_1'(t) = \frac{\phi_2(t)u(t)}{\phi_1'(t)\phi_2(t) - \phi_1(t)\phi_2'(t)}$$

and

$$v_2'(t) = \frac{-\phi_1(t)u(t)}{\phi_1'(t)\phi_2(t) - \phi_1(t)\phi_2'(t)}$$

[9] The significance of this condition is explained in Chapter 6 for the general case of nth order linear differential equations.

[10] This will be proved in Chapter 6 for a general nth order linear differential equation.

satisfy (2.27) and (2.28) simultaneously. Integrating these expressions we obtain

$$v_1(t) = V_1(t) + c_1 \quad \text{and} \quad v_2(t) = V_2(t) + c_2$$

where V_1 and V_2 are fixed antiderivatives of v_1' and v_2', and $c_1, c_2 \in \mathbf{R}$ are arbitrary constants. A solution of the non-homogeneous differential equation (2.26) is thus obtained as

$$y = \phi_1(t)V_1(t) + \phi_2(t)V_2(t) + c_1\phi_1(t) + c_2\phi_2(t) \tag{2.29}$$

We note that, as in the case of first order differential equations, the solution in (2.29) is also of the form

$$y = \phi_p(t) + \phi_c(t)$$

where

$$\phi_p(t) = \phi_1(t)V_1(t) + \phi_2(t)V_2(t)$$

is a particular solution, and

$$\phi_c(t) = c_1\phi_1(t) + c_2\phi_2(t)$$

is a complementary solution (solution of the homogeneous equation (2.24) associated with (2.26)). We will show in Chapter 6 that the expression in (2.29) includes all solutions of (2.26), and therefore, it is a general solution of (2.26).

As in the case of first order equations, we might be interested in finding among the family of solutions given by (2.29) a particular one that satisfies additional conditions. Since a general solution contains two arbitrary constants, we need two conditions to determine the values of the arbitrary constants . If these conditions involve the values of the solution and its derivative at some t_0, then the problem becomes an initial-value problem specified as

$$y'' + a_1y' + a_2y = u(t)\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_1$$

Equating the value of the general solution in (2.26) at t_0 to y_0 and the value of its derivative at t_0 to y_1, we get

$$\begin{aligned} \phi_p(t_0) + c_1\phi_1(t_0) + c_2\phi_2(t_0) &= y_0 \\ \phi_p'(t_0) + c_1\phi_1'(t_0) + c_2\phi_2'(t_0) &= y_1 \end{aligned}$$

Again linear independence of ϕ_1 and ϕ_2 guarantees that these equations can be solved for c_1 and c_2 to obtain the required solution of the initial-value problem.

Example 2.9

Solve the initial-value problem

$$y'' = 0\,, \quad y(0) = 1\,, \quad y'(0) = -1$$

The characteristic equation $s^2 = 0$ has a double root $s = 0$. Consequently,

$$\phi_1(t) = e^{0.t} = 1 \quad \text{and} \quad \phi_2(t) = te^{0.t} = t$$

are two linearly independent solutions, and a general solution is

$$y = c_1 + c_2 t$$

The initial conditions require

$$y(0) = c_1 = 1\,, \quad y'(0) = c_2 = -1$$

Thus the solution of the initial-value problem is obtained as

$$y = 1 - t$$

This problem is so simple that the solution can be found without going into the systematic procedure of finding the roots of the characteristic equation. All we have to do is to integrate y twice:

$$y'(t) = y'(0) + \int_0^t y''(\tau)\,d\tau = -1 + \int_0^t 0\,dt = -1$$

and

$$y(t) = y(0) + \int_0^t y'(\tau)\,d\tau = 1 + \int_0^t (-1)\,d\tau = 1 - t$$

Example 2.10

Let us find the unit step response of

$$y'' + py' + y = u(t)\,, \quad y(0) = y'(0) = 0 \tag{2.30}$$

for several different values of the parameter p.

We note that whatever p is, the solution for $t \le 0$ is given by $y = 0$. We are more interested in the solution for $t > 0$, for which $u(t) = 1$.

For $p = 5.2$, the characteristic equation

$$s^2 + 5.2s + 1 = 0$$

has two real roots $s = -0.2$ and $s = -5$, and a complementary solution is

$$y = c_1 e^{-0.2t} + c_2 e^{-5t}$$

To find a general solution we let

$$y = e^{-0.2t} v_1(t) + e^{-5t} v_2(t)$$

With the restriction

$$e^{-0.2t} v_1'(t) + e^{-5t} v_2'(t) = 0 \tag{2.31}$$

the derivatives of y are calculated as

$$y' = -0.2e^{-0.2t} v_1(t) - 5e^{-5t} v_2(t)$$

and

$$y'' = e^{-0.2t}(0.04v_1(t) - 0.2v_1'(t)) + e^{-5t}(25v_2(t) - 5v_2'(t))$$

Substituting y and its derivatives into the equation, we get after simplification

$$-0.2e^{-0.2t} v_1'(t) - 5e^{-5t} v_2'(t) = u(t) = 1 \tag{2.32}$$

Solving v_1' and v_2' from (2.31) and (2.32) we obtain

$$v_1'(t) = (1/4.8)e^{0.2t}\,, \quad v_2'(t) = (-1/4.8)e^{5t}$$

Hence

$$v_1(t) = (5/4.8)e^{0.2t} + c_1\,, \quad v_2(t) = -(0.2/4.8)e^{5t} + c_2$$

and a general solution for $t > 0$ is

$$\begin{aligned} y &= e^{-0.2t}((5/4.8)e^{0.2t} + c_1) + e^{-5t}(-(0.2/4.8)e^{5t} + c_2) \\ &= 1 + c_1 e^{-0.2t} + c_2 e^{-5t} \end{aligned}$$

The initial conditions

$$\begin{aligned} y(0) &= 1 + c_1 + c_2 &= 0 \\ y'(0) &= -0.2c_1 - 5c_2 &= 0 \end{aligned}$$

give $c_1 = -25/24$ and $c_2 = 1/24$. Thus the solution of the initial-value problem is obtained as

$$y = 1 - (25/24)e^{-0.2t} + (1/24)e^{-5t}\,, \quad t \geq 0$$

For $p = 2$, the characteristic equation

$$s^2 + 2s + 1 = (s+1)^2 = 0$$

has a double root $s = -1$, and a complementary solution is

$$y = c_1 e^{-t} + c_2 t e^{-t}$$

A general solution can be found by following the same steps as above. However, a simple observation allows us to avoid the burden of lengthy manipulations. We note that, for this particular problem, whatever p is, the function $\phi_p(t) = 1$ is a particular solution because

$$\phi_p''(t) + p\phi_p'(t) + \phi_p(t) = 0 + p \cdot 0 + 1 \cdot 1 = 1$$

Based on this observation we immediately write a general solution as

$$y = 1 + c_1 e^{-t} + c_2 t e^{-t}\,, \quad t \geq 0$$

Evaluating the arbitrary constants using the initial conditions, we get $c_1 = c_2 = -1$, and the solution of the initial-value problem is obtained as

$$y = 1 - (1+t)e^{-t}\,, \quad t \geq 0$$

For $p = 0.6$ the characteristic equation

$$s^2 + 0.6s + 1 = 0$$

has the complex conjugate roots $s = -\sigma \mp i\omega$, where $\sigma = 0.3$ and $\omega = \sqrt{1-\sigma^2}$. Consequently, a general solution is

$$y = 1 + c_1 e^{-\sigma t}\cos\omega\, t + c_2 e^{-\sigma t}\sin\omega\, t\,, \quad t \geq 0$$

Evaluating the arbitrary constants from the initial conditions, we obtain the solution of the initial-value problem as

$$y = 1 - e^{-\sigma t}\cos\omega\, t - \frac{\sigma}{\omega}e^{-\sigma t}\sin\omega\, t\,, \quad t \geq 0$$

It is left to the reader as an exercise to show that the solution for $p = 0$ is

$$y = 1 - \cos t\,, \quad t \geq 0$$

for $p = -0.6$ is

$$y = 1 - e^{\sigma t}\cos\omega\, t + \frac{\sigma}{\omega}\, e^{\sigma t}\sin\omega\, t\,, \quad t \geq 0$$

and for $p = -2.5$ is

$$y = 1 - (4/3)e^{0.5t} + (1/3)e^{2t}\,, \quad t \geq 0$$

The graphs of the solutions corresponding to the different values of p are shown in Figure 2.8. The reader is urged to interpret the results.

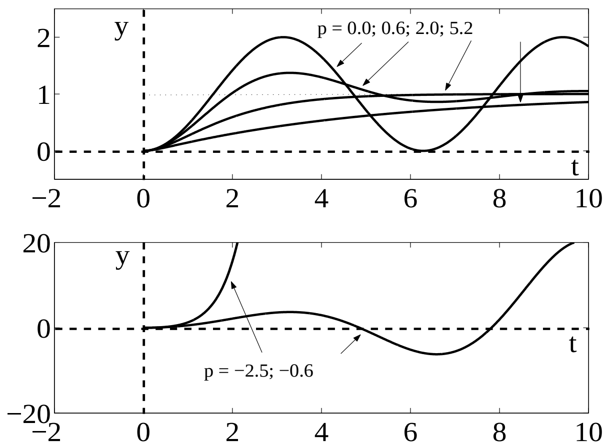

Figure 2.8: Solutions of (2.30) for several p.

2.5 Differential Operators

For a differentiable function f we use the notation f' to denote its derivative. If f' is also differentiable, then f'' denotes its second derivative, etc. An alternative notation is to denote the derivative of f by $D(f)$, where D stands for the differentiation operation. We call D the **differential operator**. Then the higher order derivatives of f can be expressed as

$$\begin{aligned} f'' &= D(f') = D(D(f)) = D^2(f) \\ f^{(3)} &= D(f'') = D(D^2(f)) = D^3(f) \end{aligned}$$

and so on, where D^2 is a short notation for the compound operator $D \circ D$, D^3 for $D \circ D^2$, etc. Each of the operators D, D^2, D^3, etc. can be viewed as a mapping from a set of functions into another such that the image of a function f under D is f', under D^2, f'', under D^3, $f^{(3)}$, etc.

Using the operator notation, an nth order linear differential equation with constant coefficients can be written as

$$(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n I)(y) = u(t) \tag{2.33}$$

where I stands for the identity operator, $I(f) = f$. Letting

$$L(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n I$$

(2.33) can be written in a compact way as

$$L(D)(y) = u(t)$$

$L(D)$ is called a **linear differential operator** with constant coefficients. Like D, D^2, etc., $L(D)$ can be viewed as a mapping that maps a function f into a combination of itself and various order derivatives.

We know from calculus that

$$D(af + bg) = aD(f) + bD(g) \tag{2.34}$$

for arbitrary functions f and g and scalars a and b. Then it is easy to show by induction on k that

$$D^k(af + bg) = aD^k(f) + bD^k(g)\,, \quad k = 1, \ldots, n$$

which in turn implies

$$L(D)(af + bg) = aL(D)(f) + bL(D)(g) \tag{2.35}$$

The significance of the property in (2.35) is that if $y = \phi(t)$ and $y = \psi(t)$ are any two solutions of the linear differential equations

$$L(D)(y) = u(t)$$

and

$$L(D)(y) = v(t)$$

then $y = a\phi(t) + b\psi(t)$ is a solution of [11]

$$L(D)(y) = au(t) + bv(t)$$

Linear differential operators with constant coefficients provide great notational and conceptual simplification because they can be treated like polynomials. For example, if $L_1(D)$ and $L_2(D)$ are two such operators then we can define a product operator $L = L_1 L_2$ such that

$$L(D)(y) = L_1(D)[L_2(D)(y)]$$

This definition allows us to factor a given linear differential operator with constant coefficients as if it were a polynomial in D.

[11] This property of linear differential equations has already been mentioned in Example 2.3.

Example 2.11

Consider a second order linear differential equation with constant coefficients

$$y'' - 3y' + 2y = 0 \tag{2.36}$$

which can be written using the operator notation as

$$(D^2 - 3D + 2I)(y) = 0$$

Treating the linear differential operator $D^2 - 3D + 2I$ like a polynomial, we can factor it out as

$$D^2 - 3D + 2I = (D - I)(D - 2I)$$

and rewrite the differential equation as

$$(D - I)[(D - 2I)(y)] = 0 \tag{2.37}$$

We can do this because

$$\begin{aligned}(D - I)[(D - 2I)(y)] &= (D - I)(y' - 2y) \\ &= D(y' - 2y) - (y' - 2y) \\ &= (y'' - 2y') - (y' - 2y) \\ &= y'' - 3y' + 2y \\ &= (D^2 - 3D + 2I)(y)\end{aligned}$$

Now letting $(D - 2I)(y) = z$, (2.37) is transformed into a first order equation

$$(D - I)(z) = z' - z = 0$$

whose solution can easily be obtained as

$$z = c_1 e^t$$

Substituting the solution back into the definition of z, we get another first order differential equation

$$(D - 2I)(y) = y' - 2y = c_1 e^t$$

Solving this final equation we obtain

$$y = c_1 e^t + c_2 e^{2t}$$

which is a general solution of (2.36).

The reader might have noticed that this is exactly what we did in finding the general solution of a second order equation whose characteristic equation has a double real root.

We will discuss the significance of linear differential operators in Chapter 3 in connection with linear transformations.

* 2.6 Further Topics on Differential Equations

2.6.1 First Order LDE with Non-Constant Coefficients

Consider a first order linear homogeneous differential equation with a non-constant coefficient

$$y' + a(t)y = 0 \tag{2.38}$$

where $a(t)$ is a given function. Writing (2.38) as

$$y'/y = -a(t)$$

and integrating both sides, we obtain

$$\ln|\,y\,| = c_1 - I(t)$$

or equivalently,

$$|\,y\,| = e^{c_1} e^{-I(t)}$$

where $I(t)$ is any antiderivative of $a(t)$. Noting that $e^{-I(t)} > 0$, we can remove the absolute value in the above expression by defining $e^{c_1} = |\,c\,|$, and thus obtain

$$y = ce^{-I(t)} \tag{2.39}$$

The expression in (2.39), which contains an arbitrary constant, is a general solution of (2.38).

Now consider the non-homogeneous equation

$$y' + a(t)y = u(t) \tag{2.40}$$

Following the method of variation of parameters, we look for a solution of the form $y = e^{-I(t)}v(t)$. Substituting y and y' into (2.40) and simplifying the equation, we get

$$v'(t) = e^{I(t)}u(t)$$

If $V(t)$ is any antiderivative of $v'(t)$ above, we have $v(t) = V(t) + c$, and a general solution of (2.40) is obtained as

$$y = e^{-I(t)}V(t) + ce^{-I(t)} = \phi_p(t) + \phi_c(t) \tag{2.41}$$

where ϕ_p is a particular solution and ϕ_c is a complementary solution.

If an initial condition $y(t_0) = y_0$ is specified, useful choices for $I(t)$ and $V(t)$ are

$$I(t) = \int_{t_0}^{t} a(\tau)\,d\tau$$

and

$$V(t) = \int_{t_0}^{t} e^{I(\tau)}u(\tau)\,d\tau = \int_{t_0}^{t} e^{\int_{t_0}^{\tau} a(\delta)\,d\delta}u(\tau)\,d\tau$$

With these choices, the solution of the initial-value problem

$$y' + a(t)y = u(t)\,, \quad y(t_0) = y_0 \tag{2.42}$$

is obtained as

$$y = e^{-\int_{t_0}^{t} a(\tau)\, d\tau}\Big(y_0 + \int_{t_0}^{t} e^{\int_{t_0}^{\tau} a(\delta)\, d\delta} u(\tau)\, d\tau\Big)$$

Defining

$$\phi(t, \tau) = e^{-\int_{\tau}^{t} a(\delta)\, d\delta} \tag{2.43}$$

and noting that

$$e^{\int_{t_0}^{\tau} a(\delta)\, d\delta} = e^{-\int_{\tau}^{t_0} a(\delta)\, d\delta} = \phi(t_0, \tau)$$

the solution above can be expressed in more compact form as

$$y = \phi(t, t_0)y_0 + \phi(t, t_0)\int_{t_0}^{t} \phi(t_0, \tau)u(\tau)\, d\tau) \tag{2.44}$$

Taking $\phi(t, t_0)$ inside the integral (it is independent of the variable of integration), and noting that (see Exercise 2.14)

$$\phi(t, t_0)\phi(t_0, \tau) = \phi(t, \tau)$$

an alternative expression for the solution is obtained as

$$y = \phi(t, t_0)y_0 + \int_{t_0}^{t} \phi(t, \tau)u(\tau)\, d\tau = \phi_o(t) + \phi_u(t) \tag{2.45}$$

Note that (2.15) is a special case of (2.45) corresponding to $\phi(t, t_0) = e^{-a(t-t_0)}$. Like (2.15), the expression in (2.45) gives the solution as the sum of two parts, one due to the initial condition y_0, and the other due to the forcing function $u(t)$.

Example 2.12

Let us solve the initial-value problem

$$y' - (\cos t)y = \cos t\,, \quad y(t_0) = y_0$$

Writing the associated homogeneous equation as

$$y'/y = \cos t$$

and integrating both sides, we obtain

$$\ln|\,y\,| = \sin t + c_1$$

or equivalently,

$$y = ce^{\sin t}$$

To find a solution of the non-homogeneous equation, we substitute $y = e^{\sin t}v(t)$ and its derivative, and obtain

$$e^{\sin t}v'(t) + (\cos t)e^{\sin t}v(t) - (\cos t)e^{\sin t}v(t) = e^{\sin t}v'(t) = \cos t$$

Thus

$$v'(t) = (\cos t)e^{-\sin t}\,, \quad v(t) = c - e^{-\sin t}$$

and a general solution is

$$y = e^{\sin t} v(t) = ce^{\sin t} - 1$$

Note that the particular solution $\phi_p(t) = -1$ could have been found by inspection.

The arbitrary constant in the general solution is evaluated using the initial condition

$$y_0 = ce^{\sin t_0} - 1 \implies c = e^{-\sin t_0}(y_0 + 1)$$

Thus the solution of the given initial-value problem is found as

$$y = e^{\sin t} e^{-\sin t_0}(y_0 + 1) - 1 = e^{\sin t - \sin t_0} y_0 + e^{\sin t - \sin t_0} - 1$$

Alternatively, we can use the formula in (2.45). Calculating

$$\phi(t, \tau) = e^{-\int_\tau^t (-\cos\delta)\, d\delta} = e^{\sin t - \sin\tau}$$

(2.45) gives the solution as

$$y = e^{\sin t - \sin t_0} y_0 + \int_{t_0}^{t} e^{\sin t - \sin\tau} \cos\tau \, d\tau = e^{\sin t - \sin t_0} y_0 + e^{\sin t - \sin t_0} - 1$$

2.6.2 Exact Equations

Nonlinear differential equations are difficult to solve even when they are first order. We now consider some special types of first order nonlinear equations for which we can find a solution.

A first order differential equation expressed in differential form

$$M(t, y)\, dt + N(t, y)\, dy = 0 \tag{2.46}$$

where M and N are given functions of two real variables t and y, defined in some rectangular region $\mathcal{D}$ of the ty plane, is said to be **exact** if there exists a function $F(t, y)$ defined and having continuous first partial derivatives in $\mathcal{D}$ such that

$$\frac{\partial F(t, y)}{\partial t} = M(t, y)\,, \quad \frac{\partial F(t, y)}{\partial y} = N(t, y) \tag{2.47}$$

Recall that the differential of a function $F(t, y)$ of two variables is

$$dF(t, y) = \frac{\partial F(t, y)}{\partial t}\, dt + \frac{\partial F(t, y)}{\partial y}\, dy$$

Thus if F satisfies the conditions in (2.47), then the equation (2.46) can be expressed as

$$M(t, y)\, dt + N(t, y)\, dy = dF(t, y) = 0$$

from which we obtain

$$F(t, y) = c$$

If this relation between t and y defines y as a differentiable function of t as $y = \phi(t)$, then ϕ is a solution of (2.46). Conversely, it can be shown that any solution $y = \phi(t)$

of the exact equation (2.46) must satisfy $F(t, \phi(t)) = c$. Such a relation between t and y is called an **implicit solution**.

Determining whether (2.46) is exact using the definition is equivalent to finding an implicit solution. Fortunately, there is a much simpler way of checking (2.46) for exactness, which also provides a constructive method to find a solution.

Suppose that equation (2.46) is exact so that there exists a function F satisfying conditions (2.47) in some rectangular region $\mathcal{D}$. If the functions M and N in (2.46) have continuous first partial derivatives in $\mathcal{D}$, then

$$\frac{\partial M(t,y)}{\partial y} = \frac{\partial^2 F(t,y)}{\partial y\,\partial t} = \frac{\partial^2 F(t,y)}{\partial t\,\partial y} = \frac{\partial N(t,y)}{\partial t}$$

On the other hand, if M and N have continuous first partial derivatives and satisfy

$$\frac{\partial M(t,y)}{\partial y} = \frac{\partial N(t,y)}{\partial t} \tag{2.48}$$

in some region $\mathcal{D}$, then the function

$$F(t,y) = \int_{t_0}^{t} M(\tau, y_0)\,d\tau + \int_{y_0}^{y} N(t,z)\,dz \tag{2.49}$$

where (t_0, y_0) is an arbitrary point in $\mathcal{D}$, satisfies (2.47) (see Exercise 2.16). Thus (2.48) gives necessary and sufficient conditions for (2.46) to be exact when the functions M and N have continuous first partial derivatives, and (2.49) provides a formula to obtain the function F when these conditions are satisfied.

Example 2.13

Show that the equation

$$(y+1)\,dt + (t-y)\,dy = 0$$

is exact, and then find a solution satisfying the initial condition $y(0) = 1$.

Since

$$\frac{\partial M(t,y)}{\partial y} = 1 = \frac{\partial N(t,y)}{\partial t}$$

everywhere, the equation is exact. Then there exists F that satisfies

$$\frac{\partial F(t,y)}{\partial t} = M(t,y) = y+1$$

Integrating both sides of this expression with respect to t, we get

$$F(t,y) = (y+1)t + f(y)$$

where f is a function to be determined. Using

$$\frac{\partial F(t,y)}{\partial y} = t + f'(y) = N(t,y) = t - y$$

we obtain

$$f'(y) = -y \implies f(y) = -y^2/2 + c_1$$

Thus

$$F(t, y) = (y + 1)t - y^2/2 + c_1$$

and an implicit solution is obtained as

$$(y + 1)t - y^2/2 + c_1 = c_2$$

or equivalently as

$$y^2 - 2ty - 2t = c$$

Alternatively, $F(t, y)$ can be obtained from the formula in (2.49) (see Exercise 2.17). The implicit solution above defines two families of solutions

$$y = t - (t^2 + 2t + c)^{1/2}$$

and

$$y = t + (t^2 + 2t + c)^{1/2}$$

No member of the first family satisfies the initial condition. Substituting the initial conditions in the expression for the second family we get $c = 1$, and the required solution is obtained as

$$y = t + (t^2 + 2t + 1)^{1/2} = 2t + 1$$

Note that, unlike linear differential equations, we cannot say that the solution above is the only solution of the initial-value problem considered.

When an equation of the form (2.46) is not exact, it is sometimes possible to find a function $I(t, y)$ such that

$$I(t, y)M(t, y)\,dt + I(t, y)N(t, y)\,dy = 0 \tag{2.50}$$

is exact (see Exercises 2.18 and 2.19). Such a function is called an **integrating factor**. If $I(t, y) \neq 0$ in a rectangular region in which equation (2.50) is exact, then (2.46) and (2.50) have the same solutions.

Example 2.14

The linear equation $y' + 2y = 0$ written in differential form as

$$2y\,dt + dy = 0$$

is not exact. Multiplying both sides by e^{2t} (which is nonzero everywhere), we get an exact equation

$$2e^{2t}y\,dt + e^{2t}\,dy = 0$$

for which F can be obtained as

$$F(t, y) = e^{2t}y + c_1$$

Thus an implicit solution is

$$e^{2t}y + c_1 = c_2$$

Letting $c = c_2 - c_1$ we get the expected explicit solution

$$y = ce^{-2t}$$

2.6.3 Separable Equations

A differential equation of the form

$$M_1(t)M_2(y)\,dt + N_1(t)N_2(y)\,dy = 0 \tag{2.51}$$

is said to be **separable**.

In a region where $N_1 \neq 0$ and $M_2 \neq 0$, (2.51) is equivalent to

$$p(t)\,dt + q(y)\,dy = 0$$

where $p = M_1/N_1$ and $q = N_2/M_2$. Integrating both sides we get an implicit solution

$$P(t) + Q(y) = c$$

where P and Q are arbitrary antiderivatives of p and q. Any function $y = \phi(t)$ that satisfies the implicit solution is an explicit solution of (2.51). In addition, if the equation $M_2(y) = 0$ has a real root

$$y = r\,, \quad r \in \mathbf{R}$$

then it is also a solution (which is lost when dividing (2.51) by $N_1(t)M_2(y)$ to obtain the equivalent equation).

Example 2.15

The equation

$$2ty^2\,dt + dy = 0$$

is separable as it can be written as

$$2t\,dt + (1/y^2)\,dy = 0$$

Integrating the last equation, we obtain an implicit solution as $t^2 - y^{-1} = c$, which defines a family of solutions

$$y = \frac{1}{t^2 - c}$$

In addition, $y = 0$ is also a solution not included in this family.

2.6.4 Reduction of Order

Consider a second order linear differential equation with non-constant coefficients described as

$$y'' + a_1(t)y' + a_2(t)y = u(t) \tag{2.52}$$

If two linearly independent solutions of the associated homogeneous equation are known (that is, if a complementary solution is known), then a general solution can be obtained by the method of variation of parameters. Unfortunately, except in special cases, there is no general method of finding a complementary solution. However, if a solution $y = \phi(t)$ of the associated homogeneous equation

$$y'' + a_1(t)y' + a_2(t)y = 0$$

is known, then a general solution of the form $y = v(t)\phi(t)$ can be obtained as follows. By substituting y and its derivatives

$$\begin{aligned} y' &= v'\phi + v\phi' \\ y'' &= v''\phi + 2v'\phi' + v\phi'' \end{aligned}$$

into (2.52), we get

$$(\phi'' + a_1\phi' + a_2\phi) + \phi v'' + (2\phi' + a_1\phi)v' = u$$

where we dropped the argument t for simplicity. Since ϕ is a solution of the associated homogeneous equation, the term in the first parenthesis above vanishes. Letting $w = v'(t)$ the equation reduces to

$$\phi(t)w' + [2\phi'(t) + a_1(t)\phi(t)]w = u(t)$$

which is a first order equation in w that can be solved by known methods. Then v is obtained by integrating w.

Example 2.16

Solve the second order differential equation

$$y'' - (3/t)y' + (4/t^2)y = 1/t\,, \quad t > 0$$

if it is given that $y = t^2$ is a solution of the associated homogeneous equation.

Letting $y = t^2 v(t)$ and substituting

$$\begin{aligned} y' &= 2tv(t) + t^2 v'(t) \\ y'' &= 2v(t) + 4tv'(t) + t^2 v''(t) \end{aligned}$$

into the given equation, we obtain after simplification

$$t^2 v''(t) + tv'(t) = 1/t$$

Defining $w = v'(t)$, the last equation reduces to a first order equation

$$w' + (1/t)w = 1/t^3$$

A solution of the last equation is found as

$$w = c_1/t - 1/t^2$$

Integrating, we get

$$v = c_1 \ln t + c_2 - 1/t$$

Thus a solution of the original problem is obtained as

$$y = c_1 t^2 \ln t + c_2 t^2 - t$$

2.7 Systems of Differential Equations

Consider an nth order differential equation of the form (2.2)

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}) \tag{2.53}$$

together with a set of n initial conditions

$$y(t_0) = y_0\,,\ y'(t_0) = y_1\,,\ \ldots\,,\ y^{(n-1)}(t_0) = y_{n-1} \tag{2.54}$$

Let us define a set of new dependent variables

$$x_1 = y\,,\ x_2 = y'\,,\ \ldots\,,\ x_n = y^{(n-1)}$$

Their derivatives can easily be obtained using the definition and (2.53) as

$$\begin{array}{lllllll} x_1' & = & y' & = & x_2 & & \\ x_2' & = & y'' & = & x_3 & & \\ & \vdots & & & & & \\ x_{n-1}' & = & y^{(n-1)} & = & x_n & & \\ x_n' & = & y^{(n)} & = & f(t, y, y', \ldots, y^{(n-1)}) & = & f(t, x_1, x_2, \ldots, x_n) \end{array} \tag{2.55}$$

The equations in (2.55) form a system of n first order differential equations which can be written in matrix form as

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x})\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2.56}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} y_0 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix}, \quad \mathbf{f}(t, \mathbf{x}) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ f(t, x_1, x_2, \ \ldots\ , x_n) \end{bmatrix}$$

and $\mathbf{x}'$ denotes element-by-element derivative of $\mathbf{x}$.

The nth order differential equation (2.53) and the system of first order differential equations in (2.56) are equivalent in the sense that there is a one-to-one correspondence between their solutions: If $y = \phi(t)$ is the solution of (2.53) corresponding to the initial conditions in (2.54), then

$$\mathbf{x} = \boldsymbol{\phi}(t) = \mathrm{col}\,[\,\phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)\,]$$

is the solution of (2.56). Conversely, if

$$\mathbf{x} = \boldsymbol{\phi}(t) = \mathrm{col}\,[\,\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\,]$$

is the solution of (2.56), then $y = \phi_1(t)$ is the solution of (2.53) that satisfies the initial conditions in (2.54). Furthermore, $\phi_1'(t) = \phi_2(t), \ldots, \phi_{n-1}'(t) = \phi_n(t)$.[12]

[12] Here we assumed that (2.53) and (2.56) have unique solutions that satisfy the given initial conditions. This is indeed the case under certain assumptions concerning the function f in (2.53). The reader is referred to Appendix B for details.

Note that if the differential equation in (2.53) is a linear one as in (2.3), then the last equation in (2.55) becomes

$$x_n' = -a_n(t)x_1 - \cdots - a_2(t)x_{n-1} - a_1(t)x_n + u(t)$$

and accordingly, the system in (2.56) takes the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{u}(t) \tag{2.57}$$

where

$$A(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_n(t) & -a_{n-1}(t) & \cdots & -a_1(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u(t) \end{bmatrix}$$

If, in addition, the linear differential equation has constant coefficients, then $A(t)$ becomes a constant matrix. (2.57) is called a system of linear differential equations. We will study systems of linear differential equations in Chapter 6.

Example 2.17

The second order differential equation

$$y'' + a_1y' + a_2y = u(t)\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_1$$

is equivalent to the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ -a_2x_1 - a_1x_2 + u(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

where $x_1 = y$ and $x_2 = y'$.

Note that since

$$\begin{aligned} \mathbf{f}(t,\mathbf{x}) &= \begin{bmatrix} x_2 \\ -a_2x_1 - a_1x_2 + u(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u(t) \end{bmatrix} = A\mathbf{x} + \mathbf{u}(t) \end{aligned}$$

the system of differential equations is linear.

Rewriting an nth order differential equation as an equivalent system of first order differential equations allows us to use some well-established numerical solution techniques as we consider in the next section.

2.8 Numerical Solution of Differential Equations

As we mentioned earlier, nonlinear differential equations and even linear equations of order higher than one with non-constant coefficients are difficult, and most of the time impossible to solve analytically. However, solutions of such equations, if they are known to exist, can be approximated with a desired degree of accuracy by using numerical techniques.

Consider an initial-value problem involving a first-order differential equation, not necessarily linear:

$$y' = f(t, y)\,, \quad y(t_0) = y_0 \tag{2.58}$$

Suppose that (2.58) has unique solution $y = \phi(t)$ on some interval $\mathcal{I}$ that includes t_0 so that

$$\phi'(t) = f(t, \phi(t))\,, \quad \phi(t_0) = y_0 \tag{2.59}$$

Consider the Taylor series expansion of ϕ about a point $t \geq t_0$ in $\mathcal{I}$.

$$\phi(t + h) = \phi(t) + h\phi'(t) + R(t, h)$$

where $R(t, h)$ denotes the remainder and is proportional to h^2. For sufficiently small h, $\phi(t + h)$ can be approximated as

$$\phi(t + h) \approx \phi(t) + hf(t, \phi(t))$$

where $\phi'(t)$ is substituted from (2.59). Let $t_k = t_0 + kh, k = 0, 1, \ldots,$ and let w_k denote the approximate value of $\phi(t_k)$. Then the approximate expression for $\phi(t+h)$ above evaluated at $t = t_k$ becomes

$$w_{k+1} = w_k + hf(t_k, w_k)\,, \quad k = 0, 1, \ldots \tag{2.60}$$

which allows us to obtain the approximate values of the solution recursively, starting with $w_0 = \phi(t_0) = y_0$. This technique of obtaining an approximate solution to an initial-value problem is known as the **Euler method**, and is suitable for computer implementation.

Note that the recursion relation in (2.60) runs forward and gives the approximate values of the solution for $t \geq t_0$. To obtain an approximate solution for $t \leq t_0$ all we have to do is to replace h with $-h$. Then with $t_k = t_0 + kh, k = 0, -1, \ldots$, the backward recursion becomes

$$w_{k-1} = w_k - hf(t_k, w_k)\,, \quad k = 0, -1, \ldots \tag{2.61}$$

Example 2.18

The initial-value problem

$$y' = -y + 1\,, \quad y(0) = 0 \tag{2.62}$$

has the exact solution

$$y = 1 - e^{-t}\,, \quad t \geq 0$$

The Euler method gives the recursion relation

$$w_{k+1} = w_k + h(1 - w_k)\,, \quad w_0 = 0$$

for the approximate solution. The following MATLAB code runs the recursion relation for $0 \leq t_k \leq 5$ with a step size of $h = 0.5$, and plots the approximate solution together with the exact solution.

```
t=0:0.01:5;                    % range of t for solution
y=1-exp(-t);                   % exact solution
```

```
h=0.5;                          % step size
k=1; tk(1)=0; wk(1)=0;          % initialization
while tk(k)<5                   % recursion
   tk(k+1)=tk(k)+h;
   wk(k+1)=(1-h)*wk(k)+h;
   k=k+1;
end
plot(t,y,tk,wk,'.')
```

Plots of the exact and approximate solutions are shown in Figure 2.9. The reader may try a different step size to observe its effect on the quality of approximation.

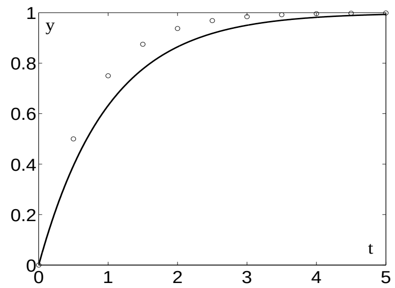

Figure 2.9: Exact and approximate solutions of (2.62).

Note that the recursion formula in (2.60) can be obtained directly from (2.58) by the following replacement of the variables.

$$\begin{aligned} t &\longleftarrow t_k \\ y &\longleftarrow w_k \\ y' &\longleftarrow \frac{1}{h}(w_{k+1} - w_k) \end{aligned} \tag{2.63}$$

This observation suggests that the Euler method can be generalized to higher order differential equations provided that we derive approximate expressions for higher derivatives of the solution. For this purpose, let us define $\psi(t) = \phi'(t)$ and denote the approximate value of $\psi(t_k) = \phi'(t_k)$ by v_k. Then

$$\begin{aligned} \phi''(t_k) = \psi'(t_k) &\approx \frac{1}{h}(v_{k+1} - v_k) \\ &\approx \frac{1}{h}[\phi'(t_{k+1}) - \phi'(t_k)] \\ &\approx \frac{1}{h}(\frac{w_{k+2} - w_{k+1}}{h} - \frac{w_{k+1} - w_k}{h}) \\ &= \frac{1}{h^2}(w_{k+2} - 2w_{k+1} + w_k) \end{aligned}$$

Thus the replacement

$$y'' \longleftarrow \frac{1}{h^2}(w_{k+2} - 2w_{k+1} + w_k) \tag{2.64}$$

in addition to those in (2.63), in a second order differential equation

$$y'' = f(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

yields the recursion relation

$$w_{k+2} = 2w_{k+1} - w_k + h^2 f(t_k, w_k, \frac{w_{k+1} - w_k}{h})$$

The initial values w_0 and w_1 needed to start the recursion are obtained from the initial conditions as

$$w_0 = y_0, \quad w_1 \approx h\phi'(t_0) + w_0 = hy_1 + y_0$$

Example 2.19

The second order initial-value problem

$$y'' + 0.6y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1 \tag{2.65}$$

is similar to the one considered in Example 2.10, and has the exact solution

$$y = \frac{1}{\omega} e^{-\sigma t} \sin \omega t$$

where $\sigma = 0.3$ and $\omega = \sqrt{1 - \sigma^2} \approx 0.9539$.

The substitutions in (2.63) and (2.64) yield

$$w_{k+2} = (2 - 0.6\,h)w_{k+1} + (-1 + 0.6\,h - h^2)w_k, \; w_0 = 0, \; w_1 = h \tag{2.66}$$

The approximate solution obtained from the recursion relation above with $h = 0.1$ is shown in Figure 2.10 together with the exact solution.

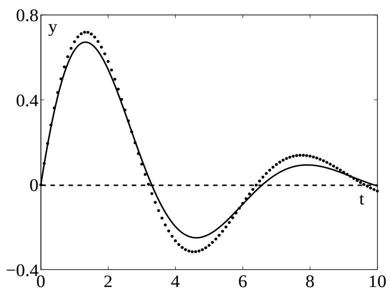

Figure 2.10: Exact and approximate solutions of (2.65).

An alternative approach to solving a high order differential equation numerically is to transform it into a system of first order differential equations as explained in the previous section. Note that a system of differential equations as in (2.56) is no different than a first order differential equation except that $\mathbf{x}$ and $\mathbf{f}$ are now column vectors rather than scalars. However, this does not make any difference in the application of the Euler method. Following the same argument leading to (2.60), a recursion relation

$$\mathbf{w}_{k+1} = \mathbf{w}_k + h\,\mathbf{f}(t_k, \mathbf{w}_k), \quad \mathbf{w}_0 = \mathbf{x}_0 \tag{2.67}$$

can be obtained for the approximate solution.

Example 2.20

With $x_1 = y, x_2 = y'$, the second order differential equation in (2.65) is transformed into

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - 0.6\,x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.68}$$

and the Euler method yields

$$\begin{bmatrix} w_{1,k+1} \\ w_{2,k+1} \end{bmatrix} = \begin{bmatrix} w_{1k} + hw_{2k} \\ (1 - 0.6\,h)w_{2k} - hw_{1k} \end{bmatrix}, \quad \begin{bmatrix} w_{10} \\ w_{20} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.69}$$

Running the recursion relation with $h = 0.1$ and plotting w_{1k} we obtain the same approximate solution as the one in Figure 2.10. Of course, this is not a coincidence, because the recursion relations in (2.66) and (2.69) are equivalent (see Exercise 2.32).

There are other numerical solution methods that are more sophisticated and more accurate than Euler method. MATLAB provides several built-in functions that use a variable step size to solve systems of first-order differential equations of the form (2.56). The reader is referred to Appendix D for a brief summary of the use of these functions.

2.9 Exercises

1. Find a general solution of the following first order linear differential equations
   (a) $y' + y = 3e^{2t} + 5\sin 2t$
   (b) $y' + 2y = 2t - 1$
   (c) $y' - y = e^t$
   (d) $y' + (1/t)y = 0$
   (e) $y' - (\cos t)y = \cos t$
2. Solve the following initial-value problems
   (a) $2y' + y = t\,, \quad y(0) = 0$
   (b) $y' + y = t^2\,, \quad y(0) = 2$
   (c) $y' + y = \begin{cases} 0, & t < 0 \quad \text{or} \quad t > 1 \\ 1, & -1 < t < 1 \end{cases}, \quad y(0) = 0$
   (d) $x^2 \dfrac{dy}{dx} + 2xy = 2\sin x\,, \quad y(2\pi) = 0$
3. Suppose that $\sigma \neq -a$.
   (a) Show that the first order differential equation

   $$y' + ay = e^{\sigma t}$$

   has a particular solution of the form $\phi_p(t) = Ae^{\sigma t}$, and find A by substitution.

   (b) Show that

   $$y' + ay = te^{\sigma t}$$

   has a particular solution of the form $\phi_p(t) = (A_0 t + A_1)e^{\sigma t}$, and find A_0 and A_1.

(c) Generalize the result in part (b) and write down the form of a particular solution of

$$y' + ay = t^m e^{\sigma t}$$

4. (a) Show that the first order differential equation

$$y' - \sigma y = e^{\sigma t}$$

has a particular solution of the form $\phi_p(t) = Ate^{\sigma t}$, and find A.

(b) Generalize the result in part (a) and write down the form of a particular solution of

$$y' - \sigma y = t^m e^{\sigma t}$$

5. (a) Find the solution $y = \phi(t)$ of the initial-value problem

$$y' + y = u(t)\,, \quad y(0) = y_0$$

where

$$u(t) = \begin{cases} 0, & t < 0 \\ \cos t, & t > 0 \end{cases}$$

(b) Show that there exists a periodic function $\phi_P(t)$ such that

$$\lim_{t \to \infty} (\phi(t) - \phi_P(t)) = 0$$

independent of y_0.

(c) Find value of y_0 such that $\phi(t) = \phi_P(t)$.

6. Suppose that the initial-value problem

$$y' + ay = u(t)\,, \quad y(0) = y_0$$

has the solution $y = \phi(t)$. Show that the solution of the initial-value problem

$$y' + ay = u(t - t_0)\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_1$$

is $y = \psi(t) = \phi(t - t_0)$. Hint: Substitute $u(t - t_0)$ for $u(t)$ in (2.15).

7. A first order differential equation of the form

$$y' + p(t)y = q(t)y^n$$

where $n \neq 1$, is called a **Bernoulli equation**.

(a) Show that a change of the dependent variable $x = y^{1-n}$ transforms the Bernoulli equation to a first order linear differential equation in x.

(b) Find the solution of the initial-value problem

$$y' + y + y^2 = 0\,, \quad y(0) = 1$$

and indicate the interval on which the solution is valid.

8. Show, by direct substitution, that if the characteristic equation of the second order linear differential equation in (2.24) has a pair of complex conjugate roots $s_{1,2} = \sigma \mp i\omega$, then each of the functions $\phi_1(t) = e^{\sigma t}\cos\omega t$ and $\phi_2(t) = e^{\sigma t}\sin\omega t$ is a solution of (2.24). Hint: Note that

$$\begin{aligned}(\sigma + i\omega)^2 &+ a_1(\sigma + i\omega) + a_2 \\ &= \sigma^2 - \omega^2 + a_1\sigma + a_2 + i(2\sigma\omega + a_1\omega) = 0\end{aligned}$$

9. Show that if $\sigma \in \mathbf{R}$ is not a root of the characteristic equation of the second order differential equation

$$y'' + a_1 y' + a_2 y = e^{\sigma t}$$

then it has a particular solution of the form $\phi_p(t) = Ae^{\sigma t}$, and find A.

10. Show that if $\sigma + i\omega$ is not a root of the characteristic equation of the second order differential equation

$$y'' + a_1 y' + a_2 y = e^{\sigma t}(p \cos \omega t + q \sin \omega t)$$

where $p, q \in \mathbf{R}$, then it has a particular solution of the form

$$\phi_p(t) = e^{\sigma t}(A \cos \omega t + B \sin \omega t)$$

and find A and B.

11. Solve the following initial-value problems
   (a) $y'' + 3y' + 2y = e^{-t}\,, \quad y(0) = y'(0) = 0$
   (b) $y'' + 2y' + 2y = 10 \cos 2t\,, \quad y(0) = -1\,, \quad y'(0) = 4$
   (c) $ty'' + 2y' + ty = 0\,, \quad y(\pi) = 0\,, \quad y'(\pi) = -1$. Hint: Let $v = ty$.

12. Find the solution $y = \phi_T(t)$ of the initial-value problem

$$y'' + 3y' + 2y = u_T(t)\,, \quad y(0) = y'(0) = 0$$

for $t > 0$, where $u_T(t)$ is the unit pulse in (2.22) with $t_0 = 0$, and investigate the behavior of the solution as $T \to 0$.

13. Find a general solution of the differential equation

$$t^2 y'' - ty' + y = 2t\,, \quad t > 0$$

if it is given that $y = t$ and $y = t \ln t$ are solutions of the associated homogeneous equation.

14. Show the identity

$$\phi(t, t_0)\phi(t_0, \tau) = \phi(t, \tau)$$

for $\phi(t, \tau)$ defined in (2.43).

15. Solve the following exact differential equations
   (a) $(3t^2 + y^2)\, dt + 2ty\, dy = 0$
   (b) $(ye^{xy} - 4x^3)\, dx + xe^{xy}\, dy = 0$

16. Differentiate $F(t, y)$ in (2.49) with respect to t and with respect to y to show that if M and N satisfy (2.48), then F satisfies (2.47). The expression in (2.49) is obtained by taking the line integral of dF from an arbitrary initial point $(t_0, y_0) \in \mathcal{D}$ to $(t, y) \in \mathcal{D}$ first along a horizontal line segment from (t_0, y_0) to (t, y_0), and then along a vertical line segment from (t, y_0) to (t, y), and noting that on the horizontal line segment

$$dF(\tau, z) = \frac{\partial F(\tau, y_0)}{\partial \tau}\, d\tau = M(\tau, y_0)\, d\tau$$

and on the vertical line segment

$$dF(\tau, z) = \frac{\partial F(t, z)}{\partial z}\, dz = N(t, z)\, dz$$

Obtain an alternative expression for F by integrating dF first along a vertical line segment from (t_0, y_0) to (t_0, y), and then along a horizontal line segment from (t_0, y) to (t, y).

17. Use formula (2.49) to obtain a function

$$F(t,y) = \int_{t_0}^{t} (y_0 + 1)\, d\tau + \int_{y_0}^{y} (t - z)\, dz$$

for the exact differential equation in Example 2.9, and show that it gives the same implicit solution.

18. For the following differential equation find integrating factors of the given form, and then solve the resulting exact differential equations.

   (a) $(t^2 + y^2)\, dt + ty\, dy = 0\,, \quad I(t,y) = f(t)$

   (b) $y\, dx + dy = 0\,, \quad I(x,y) = g(x)$

   (c) $(u^2 v + v^3)\, du - 2uv^2\, dv = 0\,, \quad I(u,v) = u^m v^n$

19. Let M_t denote $\partial M/\partial t$, M_y denote $\partial M/\partial y$, etc.

   (a) Show that if

   $$p = \frac{M_y - N_t}{N}$$

   is a function of t only, then

   $$I(t) = e^{\int p(t)\, dt}$$

   is an integrating factor for (2.46).

   (b) Use the result of part (a) to find an integrating factor for

   $$(2t - y^2)\, dt + ty\, dy = 0\,, \quad t > 0$$

   and then obtain an implicit solution.

   (c) Show that if

   $$q = \frac{N_t - M_y}{M}$$

   is a function of y only, then

   $$I(y) = e^{\int q(y)\, dy}$$

   is an integrating factor for (2.46).

   (d) Use the result of part (b) to find an integrating factor for

   $$y\, dt - (3t + y^4)\, dy = 0\,, \quad y > 0$$

   and then obtain an implicit solution.

20. Solve the following separable differential equations

   (a) $(2t - 1)y\, dt + dy = 0$

   (b) $\sin x \cos y\, dx + \cos x \sin y\, dy = 0$

   (c) $4uv\, du + (u^2 + 1)\, dv = 0$

21. Consider a second order differential equation

$$F(t, y', y'') = 0$$

in which the dependent variable y does not appear explicitly.

(a) Show that a change of the dependent variable $x = y'$ transforms the equation to a first order equation in the dependent variable x.

(b) Use the result of part (a) to solve the initial-value problem

$$y'' + y' = 1\,, \quad y(0) = y'(0) = 0$$

22. Consider a second order differential equation

$$F(y, \frac{dy}{dt}, \frac{d^2y}{dt^2}) = 0$$

in which the independent variable t does not appear explicitly.

(a) Show that a change of the variables

$$\frac{dy}{dt} = v \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$$

transforms the equation to a first order equation in the independent variable y and the dependent variable v.

(b) Use the result of part (a) to solve the initial-value problem

$$yy'' - (y')^2 = 0\,, \quad y(0) = 1\,, \quad y'(0) = -1$$

23. Find a general solution of

$$(t-1)y'' - ty' + y = 1$$

Hint: Look for a solution of the form $\phi_c(t) = At + B$ for the associated homogeneous equation.

24. Solve the initial-value problem

$$ty'' + 2y' + ty = 0\,, \quad y(\pi) = 0\,, \quad y'(\pi) = -1$$

Hint: Use a change of the dependent variable as $v = ty$.

25. Solve the initial-value problems in Exercise 2.2 by using the MATLAB function `ode23`. Plot the resulting solutions and the exact solutions obtained in Exercise 2.2 on the same graph. The MATLAB function `ode23` requires a user defined function (call it `myfunction` for future use) that evaluates the vector-valued function $\mathbf{f}(t, \mathbf{x})$ in (2.56) and returns it as a vector `xdot`.

26. Let $f(t) = \sin t$, whose derivatives are $f'(t) = \cos t$ and $f''(t) = -\sin t$. Let $w_k = f(kh) = \sin kh$ denote the value of f at $t = kh$, $k = 0, 1, \ldots$, and let

$$f'(kh) \approx \frac{w_{k+1} - w_k}{h} \quad \text{and} \quad f''(kh) \approx \frac{w_{k+2} - 2w_{k+1} + w_k}{h^2}$$

be the Euler approximations of f' and f'' at $t = kh$. Use MATLAB to compute the approximate values of f' and f'' over the interval $0 \leq t \leq 10$ using a step size of $h = 0.1$, and plot the exact and approximate values of each derivative on the same graph.

27. Write a MATLAB function

```
function [tk, wk] = myeuler(ti, t0, tf, h, x0)
```

to implement the Euler method, where `h` $= h$ is the step size, `ti` $= t_i = t_0 - Mh$ and `tf` $= t_f = t_0 + Nh$ specify the end points of the interval over which the solution is to be computed, `x0` $= \mathbf{x}_0$ is an $n \times 1$ column vector, `tk` is an $N + M + 1$ dimensional array containing the points t_k, $k = -M, \ldots, t_0, \ldots, N$, and `wk` is an $n \times M + N + 1$ matrix, the columns of which are $\mathbf{w}_k$. The function `myeuler` can use the user defined function `myfunction` in Exercise 2.25 that evaluates $\mathbf{f}(t, \mathbf{x})$. Note that `myeuler` is required to solve a given system of differential equations both forward and backward. It must return only the forward solution if $t_i = t_0$ and only the backward solution if $t_f = t_0$.

28. Solve the initial-value problems in Exercise 2.2 numerically by using the function `myeuler` written in Exercise 2.27 with $h = 0.1$ and $h = 0.5$. Plot the resulting solutions and the exact solutions obtained in Exercise 2.2 on the same graph.

29. Solve the initial-value problems in Exercise 2.11 numerically by using the Euler method. Plot both the exact and numerical solutions on the same graph.

30. Solve the following initial-value problems by using both `myeuler` and `ode23`, and plot the results on the same graph.

    (a) $$y' = t^2 + y^2\,, \quad y(0) = 1$$

    (b) $$y'' - 2ty' + y^2 = 1\,, \quad y(0) = y'(0) = 1$$

31. (a) Transform each of the initial-value problems in Exercise 2.11 into a system of first-order initial-value problems.

    (b) Use the MATLAB function `ode23` to solve the systems in part (a). Plot both the exact and numerical solutions on the same graph.

    (c) Repeat (b) using the function `myeuler` written in Exercise 2.27.

32. Show that the recursion relations in (2.66) and (2.69) are equivalent, i.e., the sequence w_{1n} produced by (2.69) is the same as the sequence w_n produced by (2.66). Hint: Use (2.69) to obtain an expression for $w_{1,n+2}$ in terms of $w_{1,n+1}$ and w_{1n}.

33. A differential equation together with additional conditions on the solution that are to be satisfied at two or more values of the independent variable is called a **boundary value problem**.

    (a) Show that the boundary value problem

    $$y'' + y = 0\,, \quad y(0) = 0\,, \quad y(\pi) = 1$$

    has no solution.

    (b) Show that the boundary value problem

    $$y'' = \lambda y\,, \quad y(0) = 0\,, \quad y(\pi) = 0$$

    has a nontrivial solution if and only if $\lambda = -n^2$ for some integer n, in which case the nontrivial solution is

    $$y = c_n \sin nt$$

34. (Application) A family of curves defined by the equation

    $$F(x, y) = c\,, \quad c \in \mathbf{R}$$

    is said to be orthogonal to a second family of curves defined by the equation

    $$G(x, y) = d\,, \quad d \in \mathbf{R}$$

    if every curve of the first family intersects every curve of the second family at right angles. (Two such families are called orthogonal trajectories of each other.) Assume that every curve of each family has a well-defined gradient at every point on the xy-plane. Orthogonality of the curves is equivalent to orthogonality of the gradients at points of intersection, which requires that

    $$G_x\,dx + G_y\,dy = -F_y\,dx + F_x\,dy$$

(a) Consider a family of concentric circles defined by the equation

$$x^2 + y^2 = c^2$$

for which $F_x = 2x$ and $F_y = 2y$ at every point (x, y). The orthogonal trajectories must then satisfy

$$-2y\,dx + 2x\,dy = 0$$

Solve the above equation to obtain an expression for the orthogonal trajectories of the family of circles. Plot both families on the same graph, and verify that the curves of the two families indeed intersect at right angles.

(b) Find the orthogonal trajectories of the family of curves defined by

$$2x^2 + y^2 = c^2$$

and plot both the given family and the orthogonal family on the same graph.

(c) Repeat (b) for the family

$$y = ce^x$$

35. (Application) The behavior of a particle of mass m in vertical motion in the air near the surface of the earth is described by the second order linear differential equation

$$my'' = -mg + f(t)$$

where $y(t)$ is the position of the particle at time t measured positive upward from the surface of the earth, mg is the downward gravitational force, and $f(t)$ represents additional external forces acting on the particle.

(a) Assuming $f(t) = 0$, find the solution corresponding to the initial conditions $y(0) = y_0$ and $y'(0) = v_0$.

(b) Find the time $t = t_f$ at which the particle falls on earth and the velocity with which it hits the ground.

(c) Repeat (a) if the air resistance is modeled as $f(t) = -ky'(t)$. Show that if y_0 is large enough, then the velocity of the particle approaches a constant limit v_∞, and find v_∞.

36. (Application) According to Malthusian growth model, a certain population increases at a rate that is proportional to its current value. If $p(t)$ represents the population at time t, then

$$p' = rp$$

where r is a constant birth rate per individual.

(a) Find an expression for $p(t)$ if $p(0) = p_0$.

(b) Find T_d such that $p(T_d) = 2p_0$. T_d is called the doubling time of the population.

(c) Show that $p(t + T_d) = 2p(t)$ for all t.

37. (Application) A more realistic population model, which takes into account the death rate as well the birth rate is the logistic population model described as

$$p' = r(1 - \frac{p}{C})p$$

where r is the birth rate, and rp/C is the death rate per individual. (The model assumes that the death rate per individual increases in direct proportion to the population due to competition.)

(a) Find an expression for $p(t)$ if $p(0) = p_0$. Plot $p(t)$ for each of the cases $0 < p_0 < C$, $p_0 = C$, and $p_0 > C$.

(b) Show that $\lim_{t \to \infty} p(t) = C$ independent of p_0. C is called the carrying capacity of the environment. (If $p > C$ at any time then $p' < 0$, and $p(t)$ decreases until it eventually reaches C. If $p < C$ then $p' > 0$, and $p(t)$ increases until it reaches C. If $p = C$ then $p' = 0$, and $p(t)$ remains stable at C.)

(c) Show that if p_0 is much smaller than C, then $p(t)$ obtained from the logistic population model can be approximated with that obtained from the Malthusian growth model for small t (that is, as long as $p(t)$ remains small compared with C).

38. (Application) According to Newton's law of cooling, the rate of change of the temperature of an object is proportional to the temperature difference between the object and its surrounding medium. Denoting the temperature of the object by $T(t)$ and that of the surrounding medium by $T_m(t)$, the law of cooling is expressed as

$$T' = k(T_m(t) - T)$$

where $k > 0$ is a constant. Assume that the $T_m(t) = T_m$ is constant, and the initial temperature of the object is $T(0) = T_0$. Find an expression for $T(t)$ for each of the cases $T_0 < T_m$, $T_0 = T_m$, and $T_0 > T_m$. Are the solutions consistent with our everyday experience?

39. (Application) Consider the electrical circuit shown in Figure 2.11. Let v_s denote the voltage supplied by the source, v_r and v_c denote the voltage drops across the resistor and the capacitor, and i denote the current flowing through the circuit. The behavior of the circuit is determined by the equations

$$C\frac{dv_c}{dt} = i\,, \quad v_r = Ri\,, \quad v_s = v_r + v_c$$

where C is the capacitance and R is the resistance.

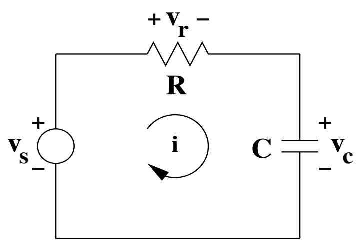

Figure 2.11: An RC circuit.

(a) Eliminate the variables v_r and i from the above equations and obtain a differential equation in the dependent variable v_c.

(b) Let $v_c(0) = 0$. Find $v_c(t)$ for $t \geq 0$ if the supply voltage $v_s(t)$ is a unit step function.

(c) Repeat (b) if $v_s(t) = \cos\omega t, t > 0$. Show that $v_c(t) \to v(t)$ as $t \to \infty$, where v is a periodic function with frequency ω (period $2\pi/\omega$).

40. (Application) Consider the mechanical system shown in Figure 2.12. The force balance on the mass requires that

$$Mx'' + Bx' + Kx = 0$$

where $x(t)$ denotes the displacement of the mass M from the equilibrium position. (Mx'' is the force accelerating the mass, Bx' represents the frictional force, and Kx is the restoring force of the spring.) Dividing the above equation by M we obtain

$$x'' + 2\zeta\omega x' + \omega^2 x = 0$$

where

$$\omega = \sqrt{\frac{K}{M}} \quad \text{and} \quad \zeta = \sqrt{\frac{MB^2}{4K}}$$

are called the natural frequency and the damping ratio of the system, respectively. Suppose that the mass is released at $t = 0$ with an initial displacement $x(0) = x_0$ and initial velocity $x'(0) = 0$. Calculate and plot $x(t)$ for $t \geq 0$ for each of the cases $\zeta = 0$, $0 < \zeta < 1$, $\zeta = 1$, and $1 < \zeta$. Show that when $\zeta = 0$, the system oscillates with natural frequency ω, and that as ζ increases, the oscillations become more and more damped, disappearing completely when $\zeta = 1$. The system is said to be undamped, underdamped, critically damped, and overdamped in the above four cases of ζ.

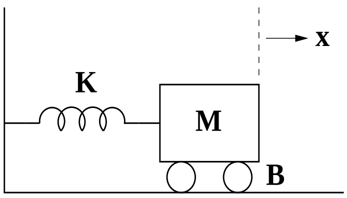

Figure 2.12: A mechanical system.

41. (Application) Rewriting a higher order differential equation in a single dependent variable as an equivalent system is not the only way we come up with a system of first order differential equations. Consider the following problem of formulating the dynamics of two competing populations, say chickens and foxes, that live in a closed environment. Assume that the chickens increase at a rate 0.4 chicken per chicken per unit time in the absence of foxes; but are killed by foxes at a rate 80 chicken per fox per unit time. Foxes, on the other hand, die of hunger at a rate 0.6 fox per fox per unit time if there are no chickens to feed upon; but when there are chickens to eat, they increase at a rate 0.003 fox per chicken per unit time. Let $x_1(t)$ and $x_2(t)$ the number of chickens and foxes at time t. Then the populations of foxes and chickens can be described by a system of two coupled first order differential equations as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0.4 & -80 \\ 0.003 & -0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

where x_{10} and x_{20} denote the initial populations of foxes and chicken at $t = 0$.

(a) Use MATLAB command `ode23` to find the solution of the system for $0 \leq t \leq 20$ for each of the following initial populations. In each case, plot $x_1/1000$ and $x_2/10$ on the same graph.

$$\mathbf{x}_0 = \begin{bmatrix} 4000 \\ 10 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 8000 \\ 40 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 8000 \\ 50 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 8000 \\ 60 \end{bmatrix}$$

Hint: The reason for plotting $x_1/1000$ and $x_2/10$ rather than x_1 and x_2 is to make the populations comparable so that x_1 does not obscure x_2 when plotted on the same graph. This scaling corresponds to counting chickens in thousands and foxes in tens, and can be done before solving the equations: Let $z_1 = x_1/1000$ and $z_2 = x_2/10$ denote the scaled populations of chickens and foxes. Rewrite the system of equations in z_1 and z_2 and observe that the coefficients in the system become comparable.

(b) Repeat (a) for

$$\mathbf{x}_0 = \begin{bmatrix} 9000 \\ 70 \end{bmatrix}$$

How can you modify your model to avoid negative population?

(c) The coefficients of the model are chosen to reflect the delicate balance of nature: If there are enough chickens initially, both populations converge to positive steady state values. Change the coefficient 0.003 to 0.0035 and solve the equation for several initial conditions. Try to interpret the result.

(d) Repeat (c) with the coefficient 0.003 changed to 0.0025.