Chapter 6
Linear Differential Equations

6.1 Systems of Linear Differential Equations

A system of n linear first order differential equations (SLDE) in n functions $x_1, \ldots, x_n$ of a real variable t has the general form

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + u_1(t) \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + u_n(t) \end{aligned} \tag{6.1}$$

where $a_{ij}(t)$ and $u_i(t)$ are given real-valued functions. The SLDE in (6.1) can be written in compact form as

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{u}(t) \tag{6.2}$$

with the obvious definitions of $\mathbf{x}, A(t)$ and $\mathbf{u}(t)$.

Suppose that $A(t)$ and $\mathbf{u}(t)$ are piecewise continuous on some interval $\mathcal{I} = (t_i, t_f)$. A vector-valued function $\boldsymbol{\phi} : \mathcal{I} \to \mathbf{R}^{n\times 1}$ is called a solution of (6.2) on $\mathcal{I}$ if

$$\boldsymbol{\phi}'(t) = A(t)\boldsymbol{\phi}(t) + \mathbf{u}(t)$$

for all $t \in \mathcal{I}$ except the discontinuity points of the right-hand side of (6.2),[1] where the derivative of $\boldsymbol{\phi} = \mathrm{col}\,[\,\phi_1, \ldots, \phi_n\,]$ is defined element-by-element as

$$\boldsymbol{\phi}' = \mathrm{col}\,[\,\phi_1', \ldots, \phi_n'\,]$$

Together with n initial conditions, $x_1(t_0) = x_{10}, \ldots, x_n(t_0) = x_{n0}$, (6.2) becomes an initial-value problem

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{u}(t)\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{6.3}$$

where

$$\mathbf{x}_0 = \mathrm{col}\,[\,x_{10}, \ldots, x_{n0}\,]$$

Then a vector-valued function $\boldsymbol{\phi}(t)$ defined on some interval $\mathcal{I}$ that includes t_0 is called a solution of (6.3) if it satisfies both the differential equation and the initial condition. The following theorem, the proof of which is given in Appendix B, presents an existence and uniqueness result about the solution of (6.3).

[1] Strictly speaking, a solution $\boldsymbol{\phi}$ must have a continuous derivative on the interval of interest. In other words, we require that $\boldsymbol{\phi} \in \mathcal{C}_1(\mathcal{I}, \mathbf{R}^{n\times 1})$. However, as we discussed in connection with Theorem 2.1, we can extend the definition of a solution to include piecewise differentiable continuous functions provided that they are continuously differentiable on every subinterval of $\mathcal{I}$ that does not contain a discontinuity point of any element of A or $\mathbf{u}$.

Theorem 6.1 *Suppose that the elements of $A(t)$ and $\mathbf{u}(t)$ are piecewise continuous on some interval $\mathcal{I}$ and $t_0 \in \mathcal{I}$. Then the initial value problem (6.3) has a unique, continuous solution $\mathbf{x} = \boldsymbol{\phi}(t)$ on $\mathcal{I}$.*

6.1.1 Homogeneous SLDE

Consider the homogeneous SLDE

$$\mathbf{x}' = A(t)\mathbf{x} \tag{6.4}$$

where elements of $A(t)$ are piecewise continuous on an interval $\mathcal{I}$. The following theorem characterizes solutions of (6.4).

Theorem 6.2 *The set of solutions of (6.4) is an n-dimensional vector space over $\mathbf{R}$.*

Proof If $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are any two solutions of (6.4), then so is $\boldsymbol{\psi} = c_1\boldsymbol{\phi}_1 + c_2\boldsymbol{\phi}_2$ for any $c_1, c_2 \in \mathbf{R}$, because

$$\boldsymbol{\psi}'(t) = c_1\boldsymbol{\phi}_1'(t) + c_2\boldsymbol{\phi}_2'(t) = c_1 A(t)\boldsymbol{\phi}_1(t) + c_2 A(t)\boldsymbol{\phi}_2(t) = A(t)\boldsymbol{\psi}(t)$$

Hence the set of solutions of (6.4) is a subspace of $\mathcal{F}(\mathcal{I}, \mathbf{R}^{n\times 1})$, and therefore, it is a vector space over $\mathbf{R}$.

Let $\mathbf{R} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be an ordered basis for $\mathbf{R}^{n\times 1}$, and let $\boldsymbol{\phi}_i(t)$ denote the unique solution of (6.4) that satisfy the initial condition $\mathbf{x}(t_0) = \mathbf{x}_i$ for some arbitrary $t_0 \in \mathcal{I}$. We claim that the set of solutions $(\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n)$ is linearly independent in $\mathcal{F}(\mathcal{I}, \mathbf{R}^{n\times 1})$.[2] To prove the claim, suppose that

$$\boldsymbol{\psi} = c_1\boldsymbol{\phi}_1 + \cdots + c_n\boldsymbol{\phi}_n = \mathbf{0}$$

Then $\boldsymbol{\psi}(t) = \mathbf{0}$ for all $t \in \mathcal{I}$, and in particular,

$$\boldsymbol{\psi}(t_0) = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$$

Since $\mathbf{R}$ is linearly independent, the last equality implies $c_1 = \cdots = c_n = 0$ proving the claim.

Let $\boldsymbol{\phi}$ be any solution of (6.4), and suppose that $\boldsymbol{\phi}(t_0) = \mathbf{x}_0$. Since $\mathbf{R}$ is a basis for $\mathbf{R}^{n\times 1}$,

$$\mathbf{x}_0 = \alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n$$

for some $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$. Let

$$\boldsymbol{\psi} = \alpha_1\boldsymbol{\phi}_1 + \cdots + \alpha_n\boldsymbol{\phi}_n$$

Then $\boldsymbol{\psi}$ is a solution of (6.4) that satisfies the initial condition

$$\mathbf{x}(t_0) = \boldsymbol{\psi}(t_0) = \alpha_1\boldsymbol{\phi}_1(t_0) + \cdots + \alpha_n\boldsymbol{\phi}_n(t_0) = \alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n = \mathbf{x}_0$$

By uniqueness of the solution of the initial-value problem consisting of (6.4) and the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, we must have

$$\boldsymbol{\phi} = \boldsymbol{\psi} = \alpha_1\boldsymbol{\phi}_1 + \cdots + \alpha_n\boldsymbol{\phi}_n$$

This shows that the solutions $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n$ also span the solution space, and therefore, they form a basis for it. We thus conclude that the solution space is n-dimensional.

[2] From now on, when talking about linear independence of a set of solutions, we will not mention the underlying vector space $\mathcal{F}(\mathcal{I}, \mathbf{R}^{n\times 1})$.

Let $\boldsymbol{\phi}_i(t), i = 1, \ldots, n$, be any n linearly independent solutions of (6.4) corresponding to a linearly independent set of initial conditions specified at some arbitrary t_0. Then the family of solutions

$$\mathbf{x} = c_1\boldsymbol{\phi}_1(t) + \cdots + c_n\boldsymbol{\phi}_n(t) \tag{6.5}$$

includes all solutions of (6.4), and therefore, characterizes a general solution. The general solution can conveniently be expressed in matrix form as

$$\mathbf{x} = X(t)\mathbf{c} \tag{6.6}$$

where

$$X(t) = [\,\boldsymbol{\phi}_1(t)\ \ \boldsymbol{\phi}_2(t)\ \cdots\ \boldsymbol{\phi}_n(t)\,]$$

and $\mathbf{c} \in \mathbf{R}^{n\times 1}$ is arbitrary. The matrix $X(t)$ is called a **fundamental matrix** of (6.4) and also of (6.2). Note that a fundamental matrix satisfies the matrix differential equation

$$X' = A(t)X \tag{6.7}$$

For convenience, we sometimes use the notation $\mathbf{x} = \boldsymbol{\phi}(t, t_0, \mathbf{x}_0)$ to denote the unique solution of (6.4) corresponding to the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. Similarly, we use the notation

$$X(t, t_0, X_0) = [\,\boldsymbol{\phi}(t, t_0, \mathbf{x}_1)\ \ \boldsymbol{\phi}(t, t_0, \mathbf{x}_2)\ \cdots\ \boldsymbol{\phi}(t, t_0, \mathbf{x}_n)\,]$$

to denote a fundamental matrix consisting of the solutions that correspond to an ordered linearly independent set of initial conditions $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ which form a nonsingular matrix

$$X_0 = [\,\mathbf{x}_1\ \ \mathbf{x}_2\ \cdots\ \mathbf{x}_n\,]$$

The special fundamental matrix corresponding to $X_0 = I$ is called the **state transition matrix**, denoted $\Phi(t, t_o)$. That is,

$$\Phi(t, t_o) = [\,\boldsymbol{\phi}(t, t_o, \mathbf{e}_1)\ \ \boldsymbol{\phi}(t, t_o, \mathbf{e}_2)\ \cdots\ \boldsymbol{\phi}(t, t_o, \mathbf{e}_n)\,]$$

where $\mathbf{e}_j$ are columns of I_n. By definition, $X(t, t_0, X_0)$ and $\Phi(t, t_o)$ are the unique solutions of the matrix differential equation (6.7) satisfying the initial conditions $X(t_0) = X_0$ and $X(t_0) = I$, respectively.

Given a fundamental matrix $X(t, t_0, X_0)$, by expressing an arbitrary initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ in terms of the columns of X_0 as $\mathbf{x}_0 = X_0\boldsymbol{\alpha}$, we observe that the solution corresponding to $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by

$$\boldsymbol{\phi}(t, t_0, \mathbf{x}_0) = X(t, t_0, X_0)\boldsymbol{\alpha} = X(t, t_0, X_0)X_0^{-1}\mathbf{x}_0 \tag{6.8}$$

In particular,

$$\boldsymbol{\phi}(t, t_0, \mathbf{x}_0) = \Phi(t, t_o)\mathbf{x}_o \tag{6.9}$$

From (6.9) we observe that the unique solution of the homogeneous SLDE in (6.4) corresponding to an initial value $\mathbf{x}(t_0) = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

The expressions in (6.8) and (6.9) imply that that any fundamental matrix can be obtained from the state transition matrix as

$$X(t,t_0,X_0)=\Phi(t,t_o)X_o$$

and conversely, given any fundamental matrix $X(t,t_0,X_0)$, we have

$$\Phi(t,t_o)=X(t,t_o,X_o)X_o^{-1}$$

These relations also imply that if $X_1(t)=X(t,t_0,X_{10})$ and $X_2(t)=X(t,t_0,X_{20})$ are any two fundamental matrices, then

$$X_2(t)=\Phi(t,t_o)X_{2o}=X_1(t)X_{1o}^{-1}X_{2o}=X_1(t)Q$$

where $Q=X_{10}^{-1}X_{20}$ is nonsingular. Conversely, if $X_1(t)=X(t,t_0,X_{10})$ is a fundamental matrix and Q is any nonsingular matrix then

$$X_2(t)=X_1(t)Q=X(t,t_0,X_{10})Q=X(t,t_0,X_{10}Q)$$

is also a fundamental matrix.

An important property of a fundamental matrix $X(t,t_0,X_0)$ is that it is nonsingular for all $t,t_0\in\mathcal{I}$. To show this suppose that $X(t_1,t_0,X_0)$ is singular for some $t_0,t_1\in\mathcal{I}$, so that

$$X(t_1,t_0,X_0)\mathbf{c}=\mathbf{0}$$

for some $\mathbf{c}\neq\mathbf{0}$. Let $\boldsymbol{\psi}(t)=X(t;t_0,X_0)\mathbf{c}$. Then $\boldsymbol{\psi}(t)$ is the solution of the initial-value problem

$$\mathbf{x}'=A(t)\mathbf{x}\,,\quad \mathbf{x}(t_1)=\mathbf{0}$$

and therefore, $\boldsymbol{\psi}(t)=\mathbf{0}$ for all $t\in\mathcal{I}$. This implies that $\boldsymbol{\psi}(t_0)=X_0\mathbf{c}=\mathbf{0}$, contradicting the fact that X_0 is nonsingular. Thus $X(t,t_0,X_0)$ is nonsingular for all $t,t_0\in\mathcal{I}$. In particular, $\Phi(t,t_o)$ is nonsingular for all $t,t_0\in\mathcal{I}$.

Another property of the state transition matrix is that

$$\Phi(t,t_1)\Phi(t_1,t_o)=\Phi(t,t_o) \tag{6.10}$$

for all $t,t_0,t_1\in\mathcal{I}$. To show this, let $\Psi(t)=\Phi(t,t_1)\Phi(t_1,t_o)$. Then

$$\frac{d}{dt}\Psi(t)=[\frac{d}{dt}\Phi(t,t_1)]\Phi(t_1,t_o)=A(t)\Phi(t,t_1)\Phi(t_1,t_o)=A(t)\Psi(t)$$

and

$$\Psi(t_1)=\Phi(t_1,t_1)\Phi(t_1,t_o)=\Phi(t_1,t_o)$$

Hence $\Psi(t)$ is the solution of the initial-value problem

$$X'=A(t)X\,,\quad X(t_1)=\Phi(t_1,t_o)$$

Obviously, $\Phi(t,t_o)$ is also a solution of this problem, and by uniqueness of the solution we must have $\Psi(t)=\Phi(t,t_1)\Phi(t_1,t_o)=\Phi(t,t_o)$.

Letting $t=t_0$ in (6.10), we obtain

$$\Phi(t_o,t_1)\Phi(t_1,t_o)=\Phi(t_o,t_o)=I$$

This gives an explicit expression for the inverse of the state state transition matrix as

$$\Phi^{-1}(t,t_o)=\Phi(t_o,t) \tag{6.11}$$

Example 6.1

Consider the system of two first order linear differential equations

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

It can be verified by substitution that

$$\boldsymbol{\phi}_1(t) = \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix}, \quad \boldsymbol{\phi}_2(t) = \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$$

are two linearly independent solutions for $-\infty < t < \infty$.

A fundamental matrix is constructed from $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ as

$$X(t) = \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix}$$

from which the state transition matrix can be obtained as

$$\begin{aligned} \Phi(t, t_o) &= X(t)X^{-1}(t_0) \\ &= \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix} \begin{bmatrix} \cos t_0 + \sin t_0 & -\sin t_0 \\ -\cos t_0 + \sin t_0 & \cos t_0 \end{bmatrix} \\ &= \begin{bmatrix} \cos(t - t_0) - \sin(t - t_0) & \sin(t - t_0) \\ -2\sin(t - t_0) & \cos(t - t_0) + \sin(t - t_0) \end{bmatrix} \end{aligned}$$

Note that $\Phi(t_o, t_o) = I$.

The solution corresponding to $\mathbf{x}(0) = \mathbf{x}_0 = \mathrm{col}\,[\,1, 0\,]$ is

$$\begin{aligned} \mathbf{x} = \Phi(t, 0)\mathbf{x}_o &= \begin{bmatrix} \cos t - \sin t & \sin t \\ -2\sin t & \cos t + \sin t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos t - \sin t \\ -2\sin t \end{bmatrix} \end{aligned}$$

and the solution corresponding to $\mathbf{x}(\pi/2) = \mathbf{x}_1 = \mathrm{col}\,[\,0, 1\,]$ is

$$\begin{aligned} \mathbf{x} = \Phi(t, \pi/2)\mathbf{x}_1 &= \begin{bmatrix} \sin(t - \pi/2) \\ \cos(t - \pi/2) + \sin(t - \pi/2) \end{bmatrix} \\ &= \begin{bmatrix} -\cos t \\ -\cos t + \sin t \end{bmatrix} \end{aligned}$$

6.1.2 Non-Homogeneous SLDE

We now turn our attention to the non-homogeneous SLDE in (6.2). Following the method of variation of parameters, we assume a solution of the form

$$\mathbf{x} = X(t)\mathbf{v}(t)$$

where $X(t)$ is any fundamental matrix, and $\mathbf{v}(t) = \mathrm{col}\,[\,v_1(t), \ldots, v_n(t)\,]$. Substituting $\mathbf{x}$ and $\mathbf{x}'$ into (6.2), we obtain after simplification

$$X(t)\mathbf{v}'(t) = \mathbf{u}(t) \tag{6.12}$$

so that

$$\mathbf{v}'(t) = X^{-1}(t)\mathbf{u}(t)$$

Hence

$$\mathbf{v}(t) = \int X^{-1}(t)\mathbf{u}(t)\,dt = \mathbf{V}(t) + \mathbf{c}$$

where $\mathbf{V}(t)$ is any antiderivative of $X^{-1}(t)\mathbf{u}(t)$, and $\mathbf{c} \in \mathbf{R}^{n\times 1}$ is arbitrary. Thus a general solution of (6.2) is obtained as

$$\mathbf{x} = X(t)(\mathbf{V}(t) + \mathbf{c}) = \boldsymbol{\phi}_p(t) + \boldsymbol{\phi}_c(t) \tag{6.13}$$

where $\boldsymbol{\phi}_p(t)$ is a particular solution, and $\boldsymbol{\phi}_c(t)$ is the complementary solution.

When an initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ is specified along with (6.2), then it is convenient to choose $X(t) = \Phi(t, t_o)$, and

$$\mathbf{V}(t) = \int_{t_0}^{t} \Phi(t_o, \tau)\mathbf{u}(\tau)\,d\tau$$

Then $\mathbf{V}(t_0) = \mathbf{0}$, and (6.13) evaluated at $t = t_0$ gives $\mathbf{c} = \mathbf{x}_0$. Thus the required solution is obtained as

$$\begin{aligned}
\mathbf{x} &= \Phi(t, t_o)(\mathbf{x}_o + \int_{t_o}^{t} \Phi(t_o, \tau)\mathbf{u}(\tau)\,d\tau) \\
&= \Phi(t, t_o)\mathbf{x}_o + \int_{t_o}^{t} \Phi(t, \tau)\mathbf{u}(\tau)\,d\tau \quad = \quad \Phi_o(t) + \Phi_u(t)
\end{aligned} \tag{6.14}$$

Note that the expressions in (6.14) are generalization of the expressions in (2.44) and (2.45) to vector case: $\Phi_o(t)$ is the part of the solution due to $\mathbf{x}_0$ and $\Phi_u(t)$ is the part due to $\mathbf{u}(t)$.

Example 6.2

Let us find the solution of

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)\,, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for a unit step input

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

When $u_1, \ldots, u_n$ in (6.1) are proportional as in this example, so that $\mathbf{u}(t) = \mathbf{b}u(t)$ for some $\mathbf{b} = \mathrm{col}\,[\,b_1, \ldots, b_n\,]$, then it is more convenient to express (6.2) as

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}u(t)$$

A fundamental matrix for the associated homogeneous SLDE has already been obtained in Example 6.1.

For $t < 0$, $u(t) = 0$ and the given SLDE reduces to a homogeneous one whose solution is $\mathbf{x} = X(t)\mathbf{c}$, where $\mathbf{c} = \mathrm{col}\,[\,c_1, c_2\,]$ is arbitrary. Using the continuity of the solution, $\mathbf{c}$ can be evaluated from the initial condition as

$$\mathbf{x}(0) = X(0)\mathbf{c} = \mathbf{x}_0 = \mathbf{0} \quad \Longrightarrow \quad \mathbf{c} = \mathbf{0}$$

Hence

$$\mathbf{x} = \mathbf{0}\,, \quad t \le 0$$

For $t > 0$, $u(t) = 1$ and the solution is of the form $\mathbf{x} = X(t)\mathbf{v}(t)$, where

$$\begin{aligned} \mathbf{v}'(t) &= X^{-1}(t)\mathbf{b}u(t) &= \begin{bmatrix} \cos t + \sin t & -\sin t \\ -\cos t + \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ & &= \begin{bmatrix} \cos t - \sin t \\ \cos t + \sin t \end{bmatrix} \end{aligned}$$

Integrating $\mathbf{v}'(t)$, we get

$$\mathbf{v}(t) = \begin{bmatrix} \sin t + \cos t + c_1 \\ \sin t - \cos t + c_2 \end{bmatrix}$$

Thus a general solution is obtained as

$$\begin{aligned} \mathbf{x} &= X(t)\mathbf{v}(t) &= \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix} \begin{bmatrix} \sin t + \cos t + c_1 \\ \sin t - \cos t + c_2 \end{bmatrix} \\ & &= \begin{bmatrix} 1 + c_1 \cos t + c_2 \sin t \\ (c_1 + c_2)\cos t + (c_2 - c_1)\sin t \end{bmatrix} \end{aligned}$$

Initial conditions give

$$\mathbf{x}(0) = \begin{bmatrix} 1 + c_1 \\ c_1 + c_2 \end{bmatrix} = \mathbf{0} \quad \Longrightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Hence

$$\mathbf{x} = \begin{bmatrix} 1 - \cos t + \sin t \\ 2\sin t \end{bmatrix}, \quad t \ge 0$$

Alternatively, using the state transition matrix Φ obtained in Example 6.1 in the expression in (6.14) with $t_0 = 0$ and $\mathbf{x}_0 = \mathbf{0}$, we get

$$\mathbf{x} = \int_0^t \Phi(t,\tau)\mathbf{b}u(\tau)\,d\tau = \int_{\mathrm{o}}^t \begin{bmatrix} \cos(t-\tau) + \sin(t-\tau) \\ 2\cos(t-\tau) \end{bmatrix} u(\tau)\,d\tau$$

For $t \le 0$, $u(\tau) = 0$ on the interval of integration, so that $\mathbf{x} = \mathbf{0}$. For $t \ge 0$, $u(\tau) = 1$ on the interval of integration and we get

$$\mathbf{x} = \begin{bmatrix} -\sin(t-\tau) + \cos(t-\tau) \\ -2\sin(t-\tau) \end{bmatrix}_{\tau=0}^{\tau=t} = \begin{bmatrix} 1 - \cos t + \sin t \\ 2\sin t \end{bmatrix}$$

6.1.3 SLDE With Constant Coefficients

As in the case of higher order linear differential equations with non-constant coefficients, it may not be possible to obtain a fundamental matrix of (6.2) when the coefficient matrix is non-constant. However, when $A(t) = A$, a constant matrix, then the state transition matrix of the SLDE in (6.4) is given explicitly by the matrix exponential function

$$\Phi(t, t_{\mathrm{o}}) = e^{A(t-t_{\mathrm{o}})} \tag{6.15}$$

This follows from the facts that

$$\frac{d}{dt}e^{A(t-t_0)} = Ae^{A(t-t_0)}$$

as shown in Example 5.21 and that

$$e^{A(t-t_0)}\,|_{\,t=t_0} = I$$

Thus the solution of the constant-coefficient-SLDE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{u}(t)\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{6.16}$$

is obtained by substituting $\Phi(t,\tau) = e^{A(t-\tau)}$ in (6.14) as

$$\mathbf{x} = e^{A(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} e^{A(t-\tau)}\mathbf{u}(\tau)\,d\tau = \boldsymbol{\phi}_o(t) + \boldsymbol{\phi}_u(t) \tag{6.17}$$

Example 6.3

The SLDE in Example 6.1 has a constant coefficient matrix

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

The eigenvalues of A are $\lambda_{1,2} = \mp i$. Let $p(s) = \alpha s + \beta$ be an interpolating polynomial for $f(s) = e^{st}$. Then from

$$\begin{array}{rcccl} i\,\alpha + \beta & = & e^{it} & = & \cos t + i\,\sin t \\ -i\,\alpha + \beta & = & e^{-it} & = & \cos t - i\,\sin t \end{array}$$

we obtain $\alpha = \sin t$ and $\beta = \cos t$. Thus

$$e^{At} = \alpha A + \beta I = \begin{bmatrix} \cos t - \sin t & \sin t \\ -2\sin t & \cos t + \sin t \end{bmatrix}$$

Hence

$$\begin{aligned} \Phi(t,t_o) &= e^{A(t-t_0)} \\ &= \begin{bmatrix} \cos(t-t_0) - \sin(t-t_0) & \sin(t-t_0) \\ -2\sin(t-t_0) & \cos(t-t_0) + \sin(t-t_0) \end{bmatrix} \end{aligned}$$

which is the same as the one constructed in Example 6.1 from a given pair of linearly independent solutions.

6.2 Modal Decomposition of Solutions

Consider a homogeneous SLDE with constant coefficients

$$\mathbf{x}' = A\mathbf{x}\,, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{6.18}$$

where we assume, without loss of generality, that $t_0 = 0$.[3] For simplicity in notation, let us denote the solution of (6.18) by $\mathbf{x} = \boldsymbol{\phi}(t,\mathbf{x}_0)$ by omitting the unnecessary argument $t_0 = 0$. Thus

$$\boldsymbol{\phi}(t,\mathbf{x}_0) = e^{At}\mathbf{x}_0 \tag{6.19}$$

[3] To study the case $t_0 \neq 0$ all we have to do is to replace t with $t - t_0$ in the expression for the state transition matrix.

6.2.1 Complex Modes

Suppose λ is an eigenvalue of A with an associated eigenvector $\mathbf{v}$, and suppose $\mathbf{x}_0 = \mathbf{v}$. Then since

$$A\mathbf{v} = \lambda\mathbf{v}$$

we have

$$e^{At}\mathbf{v} = e^{\lambda t}\mathbf{v}$$

so that the solution of (6.18) corresponding to $\mathbf{x}_0 = \mathbf{v}$ is

$$\mathbf{x} = \boldsymbol{\phi}(t, \mathbf{v}) = e^{At}\mathbf{v} = e^{\lambda t}\mathbf{v} \tag{6.20}$$

(6.20) implies that

$$\boldsymbol{\phi}(t, \mathbf{v}) \in \operatorname{span}(\mathbf{v}) \quad \text{for all } t$$

that is, a (complex) solution that starts from an eigenvector $\mathbf{v}$ remains in the direction of that eigenvector for all t.

The observation above allows us to decompose the solution starting from an arbitrary initial condition into components with specific properties. We first consider the simpler case where A has simple distinct eigenvalues λ_i with associated linearly independent eigenvectors $\mathbf{v}_i$, $i = 1, \ldots, n$. Then any arbitrary initial condition can be decomposed uniquely as

$$\mathbf{x}_0 = \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{x}_{01} + \cdots + \mathbf{x}_{0n} \tag{6.21}$$

where

$$\mathbf{x}_{0i} = \alpha_i\mathbf{v}_i$$

are components of $\mathbf{x}_0$ in one-dimensional eigenspaces $\mathbf{K}_i = \operatorname{span}(\mathbf{v}_i)$. The corresponding solution is then obtained as

$$\begin{aligned} \mathbf{x} = \boldsymbol{\phi}(t, \mathbf{x}_0) &= \alpha_1 e^{At}\mathbf{v}_1 + \cdots + \alpha_n e^{At}\mathbf{v}_n \\ &= \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + \alpha_n e^{\lambda_n t}\mathbf{v}_n \\ &= \boldsymbol{\phi}(t, \mathbf{x}_{01}) + \cdots + \boldsymbol{\phi}(t, \mathbf{x}_{0n}) \end{aligned} \tag{6.22}$$

where each

$$\boldsymbol{\phi}(t, \mathbf{x}_{0i}) = e^{\lambda_i t}\mathbf{x}_{0i} = \alpha_i e^{\lambda_i t}\mathbf{v}_i\,, \quad i = 1, \ldots, n \tag{6.23}$$

is itself a solution corresponding to an initial condition $\mathbf{x}(0) = \mathbf{x}_{0i} = \alpha_i\mathbf{v}_i$. Thus a decomposition of the initial condition $\mathbf{x}_0$ into components $\mathbf{x}_{0i}$ as in (6.20) results in a corresponding decomposition of the solution into components $\boldsymbol{\phi}(t, \mathbf{x}_{0i})$ as in (6.22). Both decompositions are a result of the fact that

$$\mathbf{C}^{n\times 1} = \mathbf{K}_1 \oplus \cdots \oplus \mathbf{K}_n$$

Each solution component $\boldsymbol{\phi}(t; \mathbf{x}_{0i})$ that starts at $t = 0$ from a point $\mathbf{x}_{0i}$ in the A-invariant subspace $\mathbf{K}_i$ and remains there for all t, is called a **mode** of the SLDE in (6.18), and the decomposition of a solution into its modes as in (6.22) is called the **modal decomposition** of solution.

Example 6.4

Let us find the solution of (6.18) for

$$A = \begin{bmatrix} -3 & -2 \\ -1 & -2 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and decompose it into its modes.

The eigenvalues and eigenvectors of A can be computed as $\lambda_1 = -4$, $\lambda_2 = -1$, and

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The initial condition can easily be decomposed into its components along the eigenvectors as

$$\mathbf{x}_0 = \mathbf{v}_1 + \mathbf{v}_2$$

Hence the solution, decomposed into its modes, is

$$\mathbf{x} = \boldsymbol{\phi}(t, \mathbf{x}_0) = e^{-4t}\mathbf{v}_1 + e^{-t}\mathbf{v}_2 = \begin{bmatrix} 2e^{-4t} - e^{-t} \\ e^{-4t} + e^{-t} \end{bmatrix}$$

The modal decomposition of the solution is illustrated in Figure 6.1. The reader should try to obtain the same solution from (6.19) by calculating the matrix function e^{At}.

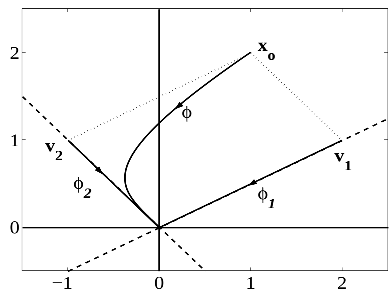

Figure 6.1: Modal decomposition of solution

We now extend the modal decomposition of solution to the general case. Let A have the characteristic polynomial

$$d(s) = \prod_{i=1}^{k} (s - \lambda_i)^{n_i}$$

where λ_i are the distinct eigenvalues with multiplicities n_i. Recall from Section 5.4 that the eigenstructure of A induces a direct sum decomposition of $\mathbf{C}^{n\times 1}$ as

$$\mathbf{C}^{n\times 1} = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{n_i} \mathcal{V}_{ij}$$

where each $\mathbf{V}_{ij}$ is an A-invariant subspace spanned by eigenvectors and/or generalized eigenvectors associated with the eigenvalue λ_i. Moreover, if the columns of the $n \times n_{ij}$ matrix P_{ij} form a basis for $\mathbf{V}_{ij}$ then

$$AP_{ij} = P_{ij} J_{ij} \tag{6.24}$$

where J_{ij} is the corresponding Jordan subblock. Since the columns of the modal matrix P constructed from P_{ij} form a basis for $\mathbf{C}^{n\times 1}$, any initial condition $\mathbf{x}_0$ can be expressed as

$$\mathbf{x}_0 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} P_{ij} \boldsymbol{\alpha}_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \mathbf{x}_{0ij} \tag{6.25}$$

for some uniquely determined $\boldsymbol{\alpha}_{ij} \in \mathbf{C}^{n_{ij}\times 1}$. Then the solution $\boldsymbol{\phi}(t;\mathbf{x}_0)$ has a corresponding decomposition

$$\boldsymbol{\phi}(t,\mathbf{x}_0) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \boldsymbol{\phi}(t,\mathbf{x}_{0ij}) \tag{6.26}$$

where

$$\boldsymbol{\phi}(t,\mathbf{x}_{0ij}) = e^{At}\mathbf{x}_{0ij} = e^{At} P_{ij} \boldsymbol{\alpha}_{ij}$$

(6.24) implies that $A^m P_{ij} = P_{ij} J_{ij}^m$ for all m, which in turn implies that $p(A)P_{ij} = P_{ij} p(J_{ij})$ for any polynomial p, and therefore, $f(A)P_{ij} = P_{ij} f(J_{ij})$ for any function f, as we have already discussed in Section 5.6. In particular,

$$\boldsymbol{\phi}(t,\mathbf{x}_{0ij}) = e^{At} P_{ij} \boldsymbol{\alpha}_{ij} = P_{ij} e^{J_{ij}t} \boldsymbol{\alpha}_{ij} \tag{6.27}$$

and (6.26) becomes

$$\boldsymbol{\phi}(t;\mathbf{x}_0) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} P_{ij} e^{J_{ij}t} \boldsymbol{\alpha}_{ij} \tag{6.28}$$

Now modes of the solution are $\boldsymbol{\phi}(t,\mathbf{x}_{0ij})$ in (6.27). As in the simple eigenvalues case, the mode $\boldsymbol{\phi}(t,\mathbf{x}_{0ij})$ starts from $\mathbf{x}_{0ij} \in \mathbf{V}_{ij}$ at $t = 0$ and remains in $\mathbf{V}_{ij}$ for all t.

Recall from Section 5.6 that

$$e^{J_{ij}t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & \cdots & t^{n_{ij}-1}/(n_{ij}-1)! \\ 0 & 1 & \cdots & t^{n_{ij}-2}/(n_{ij}-2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{6.29}$$

which implies that the mode $\boldsymbol{\phi}(t;\mathbf{x}_{0ij})$ consists of $e^{\lambda_i t}, te^{\lambda_i t}, \ldots, t^{n_{ij}-1}e^{\lambda_i t}$ terms. That is, it is of the form

$$\boldsymbol{\phi}(t,\mathbf{x}_{0ij}) = e^{\lambda_i t}\boldsymbol{\beta}_{ij1} + te^{\lambda_i t}\boldsymbol{\beta}_{ij2} + \cdots + t^{n_{ij}-1}e^{\lambda_i t}\boldsymbol{\beta}_{ijn_{ij}}$$

for some $\boldsymbol{\beta}_{ijm} \in \mathbf{C}^{n\times 1}$. Of course, not all the terms have to be present in the expression above. However, it can be shown that if the $t^q e^{\lambda_i t}$ term appears in $\boldsymbol{\phi}(t,\mathbf{x}_{0ij})$ then all the $t^m e^{\lambda_i t}, m < q$, terms must also appear (see Exercise 6.16).

When the eigenvalues of A are simple, that is, when $\nu_i = n_i = 1$ for all i, we have $P_i = P_{i1} = \mathbf{v}_i$ and $J_i = J_{i1} = \lambda_i$, where $\mathbf{v}_i$ are the eigenvectors of A associated with the eigenvalues λ_i. Then (6.25), (6.27) and (6.28) reduce to (6.21), (6.23) and (6.22) respectively.

Another case of special interest is when $\nu_i = 1$ for all eigenvalues even when they are multiple. In this case, each Jordan block consists of a single subblock, that is

$$P_i = P_{i1} = [\, \mathbf{v}_i \cdots \mathbf{v}_{in_i} \,], \quad J_i = J_{i1} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}_{n_i \times n_i}$$

where $\mathbf{v}_{i1}, \ldots, \mathbf{v}_{in_i}$ form a sequence of generalized eigenvectors associated with λ_i. Then (6.25) becomes

$$\mathbf{x}_0 = \sum_{i=1}^{k} \mathbf{x}_{0i} = \sum_{i=1}^{k} P_i \boldsymbol{\alpha}_i$$

and consequently, the modal decomposition in (6.28) takes the form

$$\boldsymbol{\phi}(t, \mathbf{x}_0) = \sum_{i=1}^{k} P_i e^{J_i t} \boldsymbol{\alpha}_i$$

Note that the ith mode consists of $e^{\lambda_i t}, te^{\lambda_i t}, \ldots, t^{n_i - 1} e^{\lambda_i t}$ terms, in general.

Example 6.5

Let us find the solution of (6.18) for the A matrix in Example 5.18, and for the initial condition $\mathbf{x}_0 = \mathrm{col}\,[\,1, 0, 0\,]$.

A modal matrix for and the Jordan form of A are already found in Example 5.18. Expressing $\mathbf{x}_0$ in terms the blocks of the modal matrix as

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} [\,1\,] = P_1 \boldsymbol{\alpha}_1 + \mathbf{v}_2 \alpha_2$$

we obtain the solution in terms of its modes as

$$\begin{aligned} \mathbf{x} &= P_1 e^{J_1 t} \boldsymbol{\alpha}_1 + \mathbf{v}_2 e^{\lambda_2 t} \alpha_2 \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \\ &= e^t \begin{bmatrix} -2t \\ -2t - 2 \\ -2t - 4 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \end{aligned}$$

6.2.2 Real Modes

If all eigenvalues of A are real, then choosing the modal matrix P also real we obtain a modal decomposition of any solution in which all the modes are real. However, if some eigenvalues of A are complex, then the eigenvectors associated with complex eigenvalues will also be complex. Although there is nothing to prevent us to treat $\mathbf{x}_0$ as an element of $\mathbf{C}^{n\times 1}$, and obtain a modal decomposition of the solution as in the previous subsection, the modes will turn out to be complex-valued in general. To decompose the solution into real components, we need to consider the complex eigenvalues and eigenvectors in conjugate pairs.

Let us first consider the case of simple eigenvalues. Suppose that A has m distinct pairs of complex conjugate eigenvalues $\lambda_{2i-1,2i} = \sigma_i \mp i\omega_i$ with associated complex conjugate pairs of eigenvectors $\mathbf{v}_{2i-1,2i} = \mathbf{u}_i \mp i\mathbf{w}_i, i = 1,\ldots,m$, and $n-2m$ real eigenvalues λ_i, with associated real eigenvectors $\mathbf{v}_i, i = 2m+1,\ldots,n$. Then, when a real initial condition vector is expressed in terms of the eigenvectors as in (6.21), it turns out that the coefficients of complex conjugate eigenvectors also appear in conjugate pairs, that is, if

$$\mathbf{x}_0 = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

then

$$\alpha_{2i-1,2i} = \beta_i \mp i\gamma_i\,, \quad i = 1,\ldots,m$$

Now consider a complex conjugate pair of solution components

$$\begin{aligned}
&\alpha_{2i-1}e^{\lambda_{2i-1}t}\mathbf{v}_{2i-1} + \alpha_{2i}e^{\lambda_{2i}t}\mathbf{v}_{2i} \\
&\quad = \; 2\,\mathrm{Re}\,\{\,\alpha_{2i-1}e^{\lambda_{2i-1}t}\mathbf{v}_{2i-1}\,\} \\
&\quad = \; 2\,\mathrm{Re}\,\{\,(\beta_i + i\gamma_i)e^{\sigma_i t}(\cos\omega_i t + i\sin\omega_i t)(\mathbf{u}_i + i\mathbf{w}_i)\,\} \\
&\quad = \; 2\,e^{\sigma_i t}(\beta_i\cos\omega_i t - \gamma_i\sin\omega_i t)\mathbf{u}_i - 2\,e^{\sigma_i t}(\gamma_i\cos\omega_i t + \beta_i\sin\omega_i t)\mathbf{w}_i
\end{aligned} \tag{6.30}$$

We observe that these two solution components, which are themselves complex solutions, add up to a real solution that lies in the two-dimensional subspace defined by the real and imaginary parts $\mathbf{u}_i$ and $\mathbf{w}_i$ of the associated eigenvectors $\mathbf{v}_{2i-1}$ and $\mathbf{v}_{2i}$, and thus define a two-dimensional mode. Repeating this for all conjugate pairs of complex modes, the solution can be decomposed as

$$\boldsymbol{\phi}(t,\mathbf{x}_0) = \sum_{i=1}^{m} \boldsymbol{\phi}_i(t) + \sum_{i=2m+1}^{n} \boldsymbol{\phi}_i(t) \tag{6.31}$$

where $\boldsymbol{\phi}_i(t)$ is of the form (6.30) for $i = 1,\ldots,m$, and of the form (6.23) for $i = 2m+1,\ldots,n$. Observe that the solution consists of $e^{\sigma_i t}\cos\omega_i t$ and $e^{\sigma_i t}\sin\omega_i t$ terms corresponding to complex eigenvalues, and $e^{\lambda_i t}$ terms corresponding to real eigenvalues.

The decomposition of the solution into real modes corresponds to direct sum decomposition of $\mathbf{R}^{n\times 1}$ as

$$\mathbf{R}^{n\times 1} = \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_m \oplus \mathbf{V}_{2m+1} \oplus \cdots \oplus \mathbf{V}_n$$

where each $\mathbf{V}_i = \text{span}(\mathbf{u}_i, \mathbf{w}_i)$, $i = 1, \ldots, m$, is a two-dimensional A-invariant subspace associated with a pair of complex conjugate eigenvalues, and each $\mathbf{V}_i = \text{span}(\mathbf{v}_i)$, $i = 2m+1, \ldots, n$, is a one-dimensional simple eigenspace associated with a real eigenvalue.

Example 6.6

Let us find the solution of (6.18) for

$$A = \begin{bmatrix} -1 & -8 & 8 \\ 8 & -1 & -8 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

and decompose it into its modes.

The eigenvalues of A are $\lambda_{1,2} = -1 \mp 8i, \lambda_3 = -1$, and the associated eigenvectors are

$$\mathbf{v}_{1,2} = \begin{bmatrix} \mp i \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Since

$$\mathbf{x}_0 = \frac{1}{2i}\mathbf{v}_1 - \frac{1}{2i}\mathbf{v}_2 + \mathbf{v}_3$$

the corresponding solution is

$$\mathbf{x} = \frac{1}{2i}\, e^{\lambda_1 t}\mathbf{v}_1 - \frac{1}{2i}\, e^{\lambda_2 t}\mathbf{v}_2 + e^{\lambda_3 t}\mathbf{v}_3$$

We observe that although A and $\mathbf{x}_0$ are both real, the modes of the solution associated with the complex eigenvalues are also complex. However, since λ_1 and λ_2 as well as $\mathbf{v}_1$ and $\mathbf{v}_2$ are conjugate pairs, then so are the corresponding modes. Hence they add up to a real solution. Indeed, the solution can be expressed as

$$\begin{aligned} \mathbf{x} &= \text{Im}\{\, e^{\lambda_1 t}\mathbf{v}_1 \,\} + e^{\lambda_3 t}\mathbf{v}_3 \\ &= \text{Im}\{\, e^{-t}(\cos 8t + i \sin 8t) \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} \} + e^{-t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= e^{-t}(\, \cos 8t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \sin 8t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) + e^{-t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \boldsymbol{\phi}_1(t) + \boldsymbol{\phi}_3(t) \end{aligned}$$

Note that $\boldsymbol{\phi}_1(t)$ lies in the two-dimensional subspace spanned by $\mathbf{u}_1 = \text{Re}\{\, \mathbf{v}_1 \,\}$ and $\mathbf{w}_1 = \text{Im}\{\, \mathbf{v}_1 \,\}$, and $\boldsymbol{\phi}_3(t)$ lies in the one-dimensional subspace spanned by $\mathbf{v}_3$. The decomposition of the solution into its real modes is illustrated in Figure 6.2.

The analysis of the general case of repeated complex eigenvalues is similar to that of the simple eigenvalue case, and is left as an exercise to the reader. It is worth to mention that when $\nu_i = 1$ for a repeated complex eigenvalue $\lambda_i = \sigma_i + i\omega_i$ with multiplicity n_i, the corresponding real mode consists of

$$e^{\sigma_i t} \cos \omega_i t, e^{\sigma_i t} \sin \omega_i t, \ldots, t^{n_i - 1} e^{\sigma_i t} \cos \omega_i t, t^{n_i - 1} e^{\sigma_i t} \sin \omega_i t$$

terms, as we illustrate by the following example.

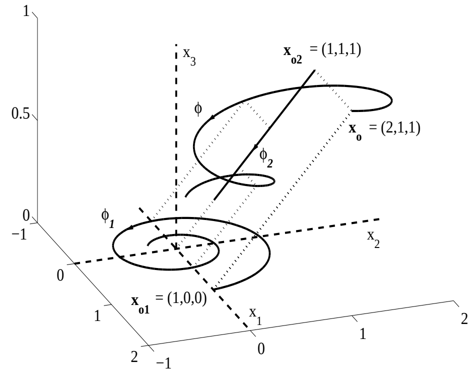

Figure 6.2: Modal decomposition of solution

Example 6.7

Let us find the modes of the solution of (6.18) for the A matrix considered in Example 5.19 and for the initial condition $\mathbf{x}_0 = \mathrm{col}\,[\,0, 1, 2, 0\,]$.

A complex modal matrix for and the Jordan form of A are already obtained in Example 5.19 as

$$P = [\,\mathbf{v}_{11} \quad \mathbf{v}_{12} \quad \mathbf{v}_{11}^* \quad \mathbf{v}_{12}^*\,] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1+i & 1 & 1-i & 1 \\ 2i & 2+2i & -2i & 2-2i \\ -2+2i & 6i & -2-2i & -6i \end{bmatrix}$$

and

$$J = P^{-1}AP = \begin{bmatrix} J_1 & O \\ O & J_1^* \end{bmatrix} = \begin{bmatrix} 1+i & 1 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 1-i & 1 \\ 0 & 0 & 0 & 1-i \end{bmatrix}$$

Decomposition of $\mathbf{x}_0$ into its components along the columns of P gives

$$\mathbf{x}_0 = \mathrm{Re}\,\{\mathbf{v}_{12}\} = \frac{1}{2}\mathbf{v}_{12} + \frac{1}{2}\mathbf{v}_{12}^* = \frac{1}{2}P_1\mathbf{e}_2 + \frac{1}{2}P_1^*\mathbf{e}_2$$

Then the corresponding solution is

$$\begin{aligned}
\mathbf{x} &= \boldsymbol{\phi}(t,\mathbf{x}_0) = \frac{1}{2}P_1 e^{J_1 t}\mathbf{e}_2 + \frac{1}{2}P_1^* e^{J_1^* t}\mathbf{e}_2 = \mathrm{Re}\,\{P_1 e^{J_1 t}\mathbf{e}_2\} \\
&= \mathrm{Re}\,\{ \begin{bmatrix} 1 & 0 \\ 1+i & 1 \\ 2i & 2+2i \\ -2+2i & 6i \end{bmatrix} \begin{bmatrix} e^{(1+i)t} & te^{(1+i)t} \\ 0 & e^{(1+i)t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \\
&= \mathrm{Re}\,\{e^t(\cos t + i\sin t)(\begin{bmatrix} 1 \\ 1+i \\ 2i \\ -2+2i \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 2+2i \\ 6i \end{bmatrix})\}
\end{aligned}$$

$$= \quad e^t \cos t \begin{bmatrix} t \\ t+1 \\ 2 \\ -2t \end{bmatrix} + e^t \sin t \begin{bmatrix} 0 \\ -t \\ -2t-2 \\ -2t-6 \end{bmatrix}$$

Note that the solution consists of $e^t \cos t$, $te^t \cos t$, $e^t \sin t$, and $te^t \sin t$ terms whose coefficient vectors span $\mathbf{R}^{4\times 1}$. No matter what the initial condition is, these four terms always appear in the solution, because $\mathbf{R}^{4\times 1}$ has no decomposition into smaller A-invariant subspaces.

6.3 nth Order Linear Differential Equations

Recall that an nth order linear differential equation (LDE) in an unknown function y of an independent variable t is of the form

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = u(t) \tag{6.32}$$

where the coefficients $a_i(t)$ and $u(t)$ are given real-valued function defined on some interval $\mathcal{I}$.

As we have already considered in Section 2.7, defining new variables

$$x_1 = y\,,\; x_2 = y'\,,\; \ldots\,,\; x_n = y^{(n-1)} \tag{6.33}$$

we obtain an equivalent system of n first order linear differential equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= -a_n(t)x_1 - a_{n-1}(t)x_2 - \cdots - a_1(t)x_n + u(t) \end{aligned}$$

which can be written in matrix form as

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{b}u(t) \tag{6.34}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix},\; A(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_n(t) & -a_{n-1}(t) & \cdots & -a_1(t) \end{bmatrix},\; \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

If $y = \phi(t)$ is a solution of (6.32), then

$$\mathbf{x} = \boldsymbol{\phi}(t) = \operatorname{col}[\,\phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)\,] \tag{6.35}$$

is a solution of (6.34). Conversely, if

$$\mathbf{x} = \boldsymbol{\phi}(t) = \operatorname{col}[\,\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\,]$$

is a solution of (6.34), then $y = \phi_1(t)$ is a solution of (6.32), and furthermore, $\phi_1'(t) = \phi_2(t), \ldots, \phi_{n-1}'(t) = \phi_n(t)$. That is, any solution of (6.34) must be of the form in (6.35). This observation allows us to apply what we already know about the solution of SLDE's to obtain solutions and their properties of LDE's.

6.3.1 Homogeneous Linear Differential Equations

Consider an nth order homogeneous LDE

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0 \tag{6.36}$$

which is transformed into a homogeneous SLDE

$$\mathbf{x}' = A(t)\mathbf{x} \tag{6.37}$$

by means of the change of variables in (6.33).

Let $X(t)$ be a fundamental matrix of (6.37) consisting of the solutions $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n$. Then it must be of the form

$$\begin{aligned} X(t) &= [\,\boldsymbol{\phi}_1(t)\ \ \boldsymbol{\phi}_2(t)\ \cdots\ \boldsymbol{\phi}_n(t)\,] \\ &= \begin{bmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & \cdots & \phi_n'(t) \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \cdots & \phi_n^{(n-1)}(t) \end{bmatrix} \end{aligned} \tag{6.38}$$

for some functions $\phi_1, \ldots, \phi_n$, each of which is a solution of (6.36). We claim that $\phi_1, \ldots, \phi_n$ are linearly independent. To prove the claim, let

$$c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n = 0$$

which means

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t) = 0 \quad \text{for all } \ t \in \mathcal{I}$$

Differentiating both sides $n-1$ times at any $t \in \mathcal{I}$, we get

$$X(t)\mathbf{c} = \mathbf{0}$$

where

$$\mathbf{c} = \operatorname{col}[\,c_1, c_2, \ldots, c_n\,]$$

Since $X(t)$ is nonsingular for all $t \in \mathcal{I}$, the last equality implies $c_1 = c_2 = \cdots = c_n = 0$, proving linear independence of $\phi_1, \ldots, \phi_n$.

Suppose that ϕ is any solution of (6.36). Then

$$\boldsymbol{\phi} = \operatorname{col}[\,\phi, \phi', \ldots, \phi^{(n-1)}\,]$$

is a solution of (6.37) and can be expressed as

$$\boldsymbol{\phi}(t) = X(t)\boldsymbol{\alpha} = \sum_{i=1}^{n} \alpha_i \boldsymbol{\phi}_i(t)$$

for some $\alpha_i \in \mathbf{R}$, where $X(t)$ is the fundamental matrix in (6.38). Considering the first element of $\boldsymbol{\phi}$, we have

$$\phi(t) = \sum_{i=1}^{n} \alpha_i \phi_i(t)$$

This shows that the set of solutions of (6.36) is an n-dimensional subspace of $\mathcal{F}(\mathcal{I}, \mathbf{R})$ having $\{ \phi_1, \phi_2, \ldots, \phi_n \}$ as a basis, and that a general solution of (6.36) is expressed as

$$y = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t) \tag{6.39}$$

Example 6.8

Consider the third order differential equation

$$y''' - 2y'' - y' + 2y = 0 \tag{6.40}$$

which is equivalent to

$$\mathbf{x}' = A\mathbf{x}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$$

The coefficient matrix A is in companion form with the characteristic polynomial

$$d(s) = s^3 - 2s^2 - s + 2 = (s-1)(s+1)(s-2)$$

Constructing a modal matrix as

$$P = [\, \mathbf{v}(\lambda_1) \;\; \mathbf{v}(\lambda_2) \;\; \mathbf{v}(\lambda_3) \,] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

we obtain

$$P^{-1}AP = D = \operatorname{diag}[\,1, -1, 2\,], \quad A = PDP^{-1}$$

Since e^{At} is a fundamental matrix, then so is

$$\begin{aligned} X(t) &= e^{At}P = Pe^{Dt} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} e^t & & \\ & e^{-t} & \\ & & e^{2t} \end{bmatrix} \end{aligned}$$

Therefore,

$$\boldsymbol{\phi}_1(t) = e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{\phi}_2(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{\phi}_3(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

are three linearly independent solutions of the equivalent system. (Note that they are of the form in (6.35).) Their first components $\phi_1(t) = e^t$, $\phi_2(t) = e^{-t}$ and $\phi_3(t) = e^{3t}$ form a basis for the solution space of (6.40). Therefore,

$$y = c_1e^t + c_2e^{-t} + c_3e^{2t}$$

is a general solution of (6.40).

Let $f_1, \ldots, f_n \in \mathcal{C}_{n-1}(\mathcal{I}, \mathbf{R})$. The matrix

$$W_{f_1,\ldots,f_n}(t) = \begin{bmatrix} f_1(t) & \cdots & f_n(t) \\ \vdots & & \vdots \\ f_1^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{bmatrix}$$

is called the **Wronski matrix** of $f_1, \ldots, f_n$, and its determinant

$$w_{f_1,\ldots,f_n}(t) = \det\ W_{f_1,\ldots,f_n}(t)$$

is called the **Wronskian** of these functions. Since every element of the Wronski matrix is continuous, $w_{f_1,\ldots,f_n} \in \mathcal{C}_0(\mathcal{I}, \mathbf{R})$.

Following the argument used in proving linear independence of the basis solutions of (6.36), we can show that

$$c_1 f_1 + \cdots + c_n f_n = 0$$

if and only if

$$W_{f_1,\ldots,f_n}(t)\mathbf{c} = \mathbf{0} \quad \text{for all } \ t \in \mathcal{I}$$

where $\mathbf{c} = \mathrm{col}\,[\,c_1, \ldots, c_n\,]$. This shows that if $w_{f_1,\ldots,f_n}(t_0) \neq 0$ for at least one $t_0 \in \mathcal{I}$, then $f_1, \ldots, f_n$ are linearly independent. The converse of this result is not true in general.[4] However, if f_i are solutions of an nth order homogeneous LDE, then the converse is also true. This follows from the fact that if $f_i = \phi_i, i = 1, \ldots, n$, are linearly independent solutions of (6.36), then their Wronski matrix is a fundamental matrix of the equivalent system of first order differential equations in (6.37), and therefore, it is nonsingular for all $t \in \mathcal{I}$.

In summary, if $\phi_1, \ldots, \phi_n$ are solutions of (6.36), then either $w_{\phi_1,\ldots,\phi_n}(t) \neq 0$ for all $t \in \mathcal{I}$, in which case $\phi_1, \ldots, \phi_n$ are linearly independent, or $w_{\phi_1,\ldots,\phi_n}(t) = 0$ for all $t \in \mathcal{I}$, in which case $\phi_1, \ldots, \phi_n$ are linearly dependent. In other words, there is no possibility of having $w_{\phi_1,\ldots,\phi_n}(t_1) \neq 0$ at some t_1 and $w_{\phi_1,\ldots,\phi_n}(t_2) = 0$ at another t_2.

6.3.2 Non-Homogeneous Linear Differential Equations

Since the non-homogeneous LDE in (6.32) is a linear equation involving a linear differential operator, it has a general solution of the form

$$y = \phi_p(t) + \phi_c(t)$$

where $\phi_p(t)$ is a particular solution, and

$$\phi_c(t) = c_1\phi_1(t) + \cdots + c_n\phi_n(t)$$

is the complementary solution consisting of the basis solutions of the associated homogeneous LDE in (6.36).

[4]For example, the functions

$$f_1(t) = t^3 \quad \text{and} \quad f_2(t) = |\,t\,|^3$$

are linearly independent on the interval $\mathcal{I} = (-1, 1)$, but their Wronskian vanishes identically on $\mathcal{I}$, that is, $w_{f_1,f_2}(t) = 0$ for all $-1 < t < 1$ (see Exercise 6.20 for details).

In Section 6.1.2 we have seen how to obtain a particular solution of a SLDE from its complementary solution by the method of variation of parameters. Since a LDE can always be transformed into a SLDE, we conclude that the method should also be applicable to an nth order LDE. (In Chapter 2, we have already used the method to solve first and second order LDE's.) The details of the method are simple and are summarized below.

Suppose that $\phi_1(t), \ldots, \phi_n(t)$ are linearly independent basis solutions of (6.36). We assume that a particular solution of (6.32) is of the form

$$\phi_p(t) = v_1(t)\phi_1(t) + \cdots + v_n(t)\phi_n(t)$$

and impose the restrictions

$$\begin{bmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & \cdots & \phi_n'(t) \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \cdots & \phi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} v_1'(t) \\ v_2'(t) \\ \vdots \\ v_n'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ u(t) \end{bmatrix} \tag{6.41}$$

on the derivatives of v_i's. Since the coefficient matrix above is a fundamental matrix of (6.37), it is nonsingular for all t, and $v_i'(t)$ can be solved uniquely. Integrating each $v_i'(t)$ we get

$$v_i(t) = \int v_i'(t)\,dt = V_i(t) + c_i$$

resulting in a general solution of (6.32)

$$y = \sum_{i=1}^{n} v_i(t)\phi_i(t) = \sum_{i=1}^{n} V_i(t)\phi_i(t) + \sum_{i=1}^{n} c_i\phi_i(t) = \phi_p(t) + \phi_c(t)$$

where $\phi_p(t)$ and $\phi_c(t)$ are a particular and the complementary solutions.

Note that the restrictions in (6.41) are equivalent to (6.12). However, to apply the method of variation of parameters to a LDE we need not transform it into an equivalent SLDE. All we need is a set of basis solutions of the associated homogeneous LDE.

Example 6.9

Consider the LDE

$$y''' - 2y'' - y' + 2y = -12e^t$$

The basis solution of the associated homogeneous LDE are obtained in Example 6.8 as

$$\phi_1(t) = e^t\,, \quad \phi_2(t) = e^{-t}\,, \quad \phi_3(t) = e^{2t}$$

Hence (6.41) becomes

$$\begin{bmatrix} e^t & e^{-t} & e^{2t} \\ e^t & -e^{-t} & 2e^{2t} \\ e^t & e^{-t} & 4e^{2t} \end{bmatrix} \begin{bmatrix} v_1'(t) \\ v_2'(t) \\ v_3'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -12e^t \end{bmatrix}$$

Solving for v_i' we get

$$v_1'(t) = 6\,, \quad v_2'(t) = -2e^{2t}\,, \quad v_3'(t) = -4e^{-t}$$

and integrating v_i's

$$v_1(t) = 6t + c_1' \,, \quad v_2'(t) = -e^{2t} + c_2' \,, \quad v_3'(t) = 4e^{-t} + c_3'$$

Thus a general solution is obtained as

$$\begin{aligned} y &= (6t + c_1')e^t + (-e^{2t} + c_2')e^{-t} + (4e^{-t} + c_3')e^{2t} \\ &= 6te^t + c_1 e^t + c_2 e^{-t} + c_3 e^{2t} \end{aligned}$$

Example 6.10

Let us find the general solution of

$$y'' - \frac{2}{t}\,y' + \frac{2}{t^2}\,y = -\frac{1}{t}\,, \quad t > 0$$

given that $\phi_1(t) = t$ and $\phi_2(t) = t^2$ are two linearly independent solutions of the associated homogeneous equation.

Writing (6.41) as

$$\begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix} \begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -1/t \end{bmatrix}$$

and solving for v_1' and v_2' we obtain

$$v_1'(t) = \frac{1}{t}\,, \quad v_2'(t) = -\frac{1}{t^2}$$

Hence

$$v_1(t) = \ln t + c_1' \,, \quad v_2(t) = \frac{1}{t} + c_2'$$

and a general solution is

$$y = (\ln t + c_1')t + (\frac{1}{t} + c_2')t^2 = t\ln t + c_1 t + c_2 t^2$$

6.4 Homogeneous LDE With Constant Coefficients

When the coefficients of the LDE (6.36) are constant, it takes the form

$$L(D)(y) = (D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n I)(y) = 0 \qquad (6.42)$$

In this case, the equivalent SLDE in (6.37) becomes

$$\mathbf{x}' = A\mathbf{x} \qquad (6.43)$$

where A is a constant matrix having the companion form in (5.29). We immediately observe that the characteristic polynomial of A is

$$d(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = L(s)$$

Incidentally, $L(s)$ is called the **characteristic polynomial** associated with the linear differential operator $L(D)$, and the equation

$$L(s) = 0 \qquad (6.44)$$

is called the **characteristic equation** of the LDE in (6.42).

Recall from Section 6.2 that if the characteristic polynomial of A is factored as

$$L(s) = d(s) = \prod_{i=1}^{k}(s-\lambda_i)^{n_i} \tag{6.45}$$

then any solution of (6.43) can be decomposed into complex modes as

$$\mathbf{x} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} t^{j-1}e^{\lambda_i t}\boldsymbol{\beta}_{ij}$$

for some $\boldsymbol{\beta}_{ij} \in \mathbf{C}^{n\times 1}$. Then, as we discussed in Section 6.3.1, the complex-valued functions

$$\psi_{ij}(t) = t^{j-1}e^{\lambda_i t}\,, \quad i = 1,\ldots,k\,;\; j = 1,\ldots,n_i \tag{6.46}$$

are linearly independent and form a basis for the complex solution space of (6.42).

If λ_i are all real, then so are $\phi_{ij} = \psi_{ij}$, and therefore, they form a basis for the real solution space as well. To find a basis for the real solution space when some λ_i are complex, suppose that $L(s)$ has m distinct pairs of complex conjugate zeros $\lambda_{2i-1,2i} = \sigma_i \mp i\omega_i, i = 1,\ldots,m$, and $k-2m$ distinct real zeros $\lambda_i, i = 2m+1,\ldots,k$, with multiplicities n_i. Then in addition to the real solutions

$$\phi_{ij}(t) = t^{j-1}e^{\lambda_i t}\,, \quad i = 2m+1,\ldots,k$$

the real and imaginary parts

$$\phi_{2i-1,j}(t) = t^{j-1}e^{\sigma_i t}\cos\omega_i t\,, \quad \phi_{2i,j}(t) = t^{j-1}e^{\sigma_i t}\sin\omega_i t\,, \quad i = 1,\ldots,m$$

of the complex solutions $\psi_{ij} = t^{j-1}e^{\lambda_i t}$ are also real solutions. Since ψ_{ij} are linearly independent then so are ϕ_{ij} (see Example 3.15), and constitute a basis for the real solution space.

The result obtained above can also be reached without reference to the equivalent SLDE in (6.43). Noting that a factorization of $L(s)$ as in (6.45) corresponds to a factorization of $L(D)$ so that (6.42) can formally be written as[5]

$$\Big[\prod_{p=1}^{k}(D-\lambda_p)^{n_p}\Big](y) = 0$$

It can be shown (see Exercise 6.23) that if $p > q$ then

$$(D-\lambda)^p(t^q e^{\lambda t}) = 0$$

Thus for $j \le n_i$

$$\Big[\prod_{p=1}^{k}(D-\lambda_p)^{n_p}\Big](t^{j-1}e^{\lambda_i t}) = \Big[\prod_{\substack{p=1\\p\neq i}}^{k}(D-\lambda_p)^{n_p}\Big](D-\lambda_i)^{n_i}(t^{j-1}e^{\lambda_i t}) = 0$$

that is, each ψ_{ij} in (6.46) is a complex solution of (6.42).

[5]Here the differential operator $D-\lambda I$ is denoted as $D-\lambda$ to simplify the notation.

Example 6.11

The second order homogeneous LDE

$$y'' + 2y' + 5y = 0$$

has the characteristic equation

$$s^2 - 2s + 5 = 0$$

with a simple pair of complex conjugate roots $\lambda_{1,2} = -1 \mp 2i$. Then

$$\phi_1(t) = e^{-t}\cos 2t \quad \text{and} \quad \phi_2(t) = e^{-t}\sin 2t$$

form a basis for the solution space, and a general solution is obtained as

$$y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$$

Example 6.12

Let us find a general solution of the homogeneous LDE

$$(D-2)^2(D^2 - 2D + 2)^2(y) = 0$$

Since the characteristic polynomial $L(s)$ is already in factored form, having a real zero $\lambda_1 = 2$ with multiplicity $n_1 = 2$, and a pair of complex conjugate zeros $\lambda_{2,3} = 1 \mp i$, with multiplicities $n_{2,3} = 2$, we write down a general solution as

$$y = c_1 e^{2t} + c_2 t e^{2t} + c_3 e^t \cos t + c_4 t e^t \cos t + c_5 e^t \sin t + c_6 t e^t \sin t$$

6.5 The Method of Undetermined Coefficients

Consider a non-homogeneous LDE with constant coefficients

$$L(D)(y) = u(t) \tag{6.47}$$

We already know that once a basis for the solution space of the associated homogeneous LDE is obtained, then a particular solution of (6.47) can be found by the method of variation of parameters. The **method of undetermined coefficients** is an alternative and practical method to obtain a particular solution when u is a linear combination of functions of the form

$$u(t) = (p_0 t^r + \cdots + p_r)e^{\sigma t} \tag{6.48}$$

or of the form

$$u(t) = (p_0 t^r + \cdots + p_r)e^{\sigma t}\cos \omega t + (q_0 t^r + \cdots + q_r)e^{\sigma t}\sin \omega t \tag{6.49}$$

where $p_0, \ldots, p_r, q_0, \ldots, q_r, \sigma, \omega$ are real constants. Such functions include polynomials, exponential functions, trigonometric functions, and their products. For example, $u(t) = t^2 - 5$ is of the form of (6.48) with $\sigma = 0$; $u(t) = te^{2t}$ is of the same form with $\sigma = 2$; $u(t) = -2\sin 5t$ is of the form of (6.49) with $\sigma = 0$ and $\omega = 5$; and $u(t) = te^t \cos t - 3e^t \sin t$ is of same form with $\sigma = \omega = 1$.

We first consider the case where $u(t)$ of the form in (6.48). Then it is a solution of a homogeneous LDE

$$L_1(D)u(t) = 0 \tag{6.50}$$

where

$$L_1(D) = (D - \sigma)^{r+1} = (D - \lambda)^{r+1}\,, \quad \lambda = \sigma \tag{6.51}$$

Suppose that $y = \phi_p(t)$ is a particular solution of (6.47), that is,

$$L(D)\phi_p(t) = u(t)$$

Operating on both sides of this equation with $L_1(D)$ and using (6.50), we get

$$L_1(D)L(D)\phi_p(t) = 0 \tag{6.52}$$

which shows that ϕ_p is a solution of the homogeneous LDE in (6.52).

Let $L(s)$ have a factorization as in (6.45).

If $\lambda = \sigma$ in (6.51) is not a zero of $L(s)$, then it appears as a new zero of $L_1(s)L(s)$, that is, $L_1(s)L(s)$ has a factorization

$$L_1(s)L(s) = (s - \sigma)^{r+1}\prod_{i=1}^{k}(s - \lambda_i)^{n_i}$$

Then ϕ_p must be of the form

$$\phi_p(t) = (A_0 t^r + \cdots + A_r)e^{\sigma t} + \sum_{i=1}^{k}\sum_{j=1}^{n_i} A_{ij}\psi_{ij}(t)$$

where ψ_{ij} are the complex solutions of the associated homogeneous LDE, and are given by (6.46). However, since they can be included in the complementary solution, ϕ_p can be assumed to have the form

$$\phi_p(t) = (A_0 t^r + \cdots + A_r)e^{\sigma t} \tag{6.53}$$

Note that ϕ_p has the same structure as u. Once we decide on the form of ϕ_p, its coefficients $A_0, \ldots, A_r$ can be determined by substituting it into (6.47), and equating the coefficients of the like terms on both sides of the resulting equation.

If $\lambda = \sigma$ in (6.51) is a zero of $L(s)$, that is, if $\sigma = \lambda_p$ for some p, then $L_1(s)L(s)$ has a factorization

$$L_1(s)L(s) = (s - \sigma)^{n_p+r+1}\prod_{\substack{i=1\\ i\neq p}}^{k}(s - \lambda_i)^{n_i}$$

Then ϕ_p must be of the form

$$\begin{aligned}
\phi_p(t) \quad &= \quad (A_0 t^{n_p+r} + \cdots + A_r t^{n_p} + A_{p1}t^{n_p-1} + \ldots + A_{pn_p})e^{\sigma t} \\
&\qquad + \sum_{\substack{i=1\\ i\neq p}}^{k}\sum_{j=1}^{n_i} A_{ij}\psi_{ij}(t) \\
&= \quad t^{n_p}(A_0 t^r + \cdots + A_r)e^{\sigma t} + \sum_{i=1}^{k}\sum_{j=1}^{n_i} A_{ij}\psi_{ij}(t)
\end{aligned}$$

As before, ψ_{ij} can be included in the complementary solution, and ϕ_p can be assumed to have the form

$$\phi_p(t) = t^{n_p}(A_0 t^r + \cdots + A_r)e^{\sigma t} \tag{6.54}$$

Comparing with (6.53), we observe that when σ is a root of the characteristic equation with multiplicity $m = n_p$, then the assumed solution in (6.53) is modified by multiplying it with t^m. As before, the coefficients $A_0, \ldots, A_r$ of the assumed particular solution can be determined by substituting it into (6.47), and equating the coefficients of the like terms on both sides of the resulting equation.

We now consider the case where $u(t)$ of the form in (6.49). Then it is a solution of a homogeneous LDE

$$L_2(D)u(t) = 0 \tag{6.55}$$

where

$$L_2(D) = (D^2 - 2\sigma D + \sigma^2 + \omega^2)^{r+1} = (D - \lambda)^{r+1}(D - \lambda^*)^{r+1} \tag{6.56}$$

with $\lambda = \sigma + i\omega$. Following the same argument as in the previous case, we can show (see Exercise 6.26) that if $\lambda = \sigma + i\omega$ is not a root of the characteristic equation $L(s) = 0$, then a particular solution of (6.47) is of the form

$$\phi_p(t) = (A_0 t^r + \cdots + A_r)e^{\sigma t}\cos\omega t + (B_0 t^r + \cdots + B_r)e^{\sigma t}\sin\omega t$$

and if $\lambda = \sigma + i\omega$ is a root of the characteristic equation $L(s) = 0$ with multiplicity m, then a particular solution is of the form

$$\phi_p(t) = t^m(A_0 t^r + \cdots + A_r)e^{\sigma t}\cos\omega t + t^m(B_0 t^r + \cdots + B_r)e^{\sigma t}\sin\omega t$$

In either case, the coefficients $A_0, \ldots, A_r$ and $B_0, \ldots, B_r$ are found by substituting the assumed particular solution into (6.47) and equating the coefficients of the like terms.

Finally, we note that if u is the sum of functions of the form in (6.48) or (6.49) with different σ's and/or ω's, then as a result of the linearity of $L(D)$ the assumed particular solution will be the sum of the corresponding assumed solutions.

Example 6.13

Let us find a general solution of the second order LDE

$$y'' + 2y' + 5y = 16te^t$$

The roots of the characteristic equation have already been obtained in Example 6.11 as $\lambda_{1,2} = -1 \mp 2i$. $u(t) = 16te^t$ is of the form in (6.48) with $\sigma = 1$, which is not a root of the characteristic equation. Consequently, we assume a particular solution of the form

$$\phi_p(t) = (A_0 t + A_1)e^t$$

Substituting ϕ_p and its derivatives

$$\phi'(t) = (A_0 t + A_0 + A_1)e^t\,, \quad \phi''(t) = (A_0 t + 2A_0 + A_1)e^t$$

into the given LDE, cancelling out the nonzero terms e^t on both sides of the equation, and arranging the terms, we get

$$8A_0 t + (4A_0 + 8A_1) = 16t$$

Equating the coefficients of the like terms and solving for A_0 and A_1, we obtain $A_0 = 2$ and $A_1 = -1$. Hence

$$\phi_p(t) = (2t-1)e^t$$

Together with the complementary solution obtained in Example 6.11, we get a general solution as

$$y = \phi_p(t) + \phi_c(t) = (2t-1)e^t + c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$$

Example 6.14

Consider the LDE

$$(D-1)^2(D-2)(y) = 6te^t$$

$u(t) = 6te^t$ is of the form in (6.48) with $\sigma = 1$, which is a root of the characteristic equation with multiplicty $m = 2$. Consequently, we assume a particular solution of the form

$$\phi_p(t) = t^2(A_0 t + A_1)e^t = (A_0 t^3 + A_1 t^2)e^t$$

Evaluating

$$\begin{aligned}
(D-1)\phi_p(t) &= (D-1)[(A_0t^3 + A_1t^2)e^t] \\
&= (3A_0t^2 + 2A_1t)e^t \\
(D-1)^2\phi_p(t) &= (D-1)[(3A_0t^2 + 2A_1t)e^t] \\
&= (6A_0t + 2A_1)e^t
\end{aligned}$$

and

$$\begin{aligned}
L(D)\phi_p(t) &= (D-2)(D-1)^2\phi_p(t) \\
&= (D-2)[(6A_0t + 2A_1)e^t] \\
&= (-6A_0t + 6A_0 - 2A_1)e^t \quad = \quad 6te^t
\end{aligned}$$

we obtain $A_0 = -1$ and $A_1 = -3$. A general solution can then be written as

$$y = t^2(-t-3)e^t + c_1te^t + c_2e^t + c_3e^{2t} = (-t^3 - 3t^2 + c_1t + c_2)e^t + c_3e^{2t}$$

Example 6.15

Let us find a general solution of the LDE

$$y'' + 2y' + 5y = 10\cos t$$

The right-hand side of the LDE is of the form in (6.49) with $\lambda = \sigma + i\omega = i$. Since $\lambda = i$ is not a root of the characteristic equation, we assume a particular solution of the form

$$\phi_p(t) = A\cos t + B\sin t$$

Substituting $\phi_p(t)$ together with

$$\phi_p'(t) = B\cos t - A\sin t\,, \quad \phi_p''(t) = -A\cos t - B\sin t$$

into the given equation and collecting the terms, we obtain

$$(4A + 2B)\cos t + (4B - 2A)\sin t = 10\cos t$$

and equating the coefficients of the like terms,

$$4A + 2B = 10\,, \quad 4B - 2A = 0$$

Solving for A and B, we get $A = 2$, $B = 1$. Thus a particular solution is obtained as

$$y_p = 2\cos t + \sin t$$

Together with the complementary solution obtained in Example 6.11, a general solution can be written as

$$y = 2\cos t + \sin t + c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$$

Let us now consider the same LDE with a different right-hand side:

$$y'' + 2y' + 5y = 16te^{-t}\sin 2t$$

Now, the right-hand side of the LDE is of the form in (6.49) with $\lambda = \sigma + i\omega = -1 + i$, which is a simple root of the characteristic equation. Then we assume a particular solution of the form

$$\phi_p(t) = t(A_0 t + A_1)e^{-t}\cos 2t + t(B_0 t + B_1)e^{-t}\sin 2t$$

It is left to the reader to show that after evaluating the coefficients of the assumed solution by substitution we get

$$\phi_p(t) = te^{-t}(-2t\cos 2t + \sin 2t)$$

6.6 Exercises

1. Show that
$$Y(t) = \begin{bmatrix} \cos t + \sin t & \sin t - \cos t \\ 2\cos t & 2\sin t \end{bmatrix}$$
is also a fundamental matrix of the SLDE in Example 6.1. How is it related to the fundamental matrix $X(t)$ found in the same example?

2. Find the solution of the initial value problem considered in Example 6.2 for a unit pulse input
$$u(t) = \begin{cases} 1/T, & 0 < t < T \\ 0, & t < 0 \ \text{ or } \ t > T \end{cases}$$
and investigate the behavior of solution as $T \to 0$.

3. (a) Show that if $A(t)$ and $A(\tau)$ commute for all $t, \tau \in \mathcal{I}$ then a fundamental matrix of (6.4) is given by the formula
$$X(t) = e^{B(t)}\,, \quad B(t) = \int A(t)\,dt$$
Hint: First show that if $A(t)$ and $A(\tau)$ commute then $B(t)$ and $B'(t)$ commute. Use this result to show that
$$\frac{d}{dt}\,B^m(t) = B'(t)B^{m-1}(t) = A(t)B^{m-1}(t)\,, \quad m = 1, 2, \ldots$$
Finally, consider
$$X(t) = e^{B(t)} = \sum_{m=0}^{\infty} \frac{1}{m!}\,B^m(t)$$

(b) Show that if $A(t) = a(t)A$ then $A(t)$ and $A(\tau)$ commute, so that

$$X(t) = e^{b(t)A}\,, \quad b(t) = \int a(t)\,dt$$

4. Find the state transition matrix for

$$A(t) = \begin{bmatrix} -t & 1 \\ -1 & -t \end{bmatrix}$$

Hint: Use the result of Exercise 6.3.

5. Find the solution of

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

corresponting to the initial conditions

(a) $x_1(0) = x_2(0) = 1$

(b) $x_1(1) = 1\,, \quad x_2(1) = 0$

6. Consider the SLDE

$$\begin{aligned} x_1' &= (\cos t)x_2 \\ x_2' &= -(\cos t)x_1 + u(t) \end{aligned}$$

(a) Obtain a fundamental matrix. Hint: Use the result of Exercise 6.3.

(b) Find the solution for $x_1(0) = x_2(0) = 0$ and $u(t) = \cos t$.

7. Obtain the solutions of the SLDE's in Exercises 6.5 and 6.6 numerically using the MATLAB function `ode23`. In each case plot the exact and numerical solutions on the same graph.

8. Consider the SLDE

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $a(t)$ is a periodic, piecewise constant function given as

$$a(t) = \begin{cases} 1, & 2k\pi < t < (2k+1)\pi \\ 0, & (2k+1)\pi < t < 2(k+1)\pi \end{cases}$$

(a) Calculate $\Phi(t,0)$ for $0 \le t \le 4\pi$. Hint: Note that

$$A(t) = \begin{cases} A_1, & 0 < t < \pi \quad \text{and} \quad 2\pi < t < 3\pi \\ A_2, & \pi < t < 2\pi \quad \text{and} \quad 3\pi < t < 4\pi \end{cases}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, $\Phi(t,0) = e^{A_1 t}$ for $0 \le t \le \pi$, $\Phi(t,0) = \Phi(t,\pi)\Phi(\pi,0) = e^{A_2(t-\pi)}e^{A_1\pi}$ for $\pi \le t \le 2\pi$, etc.

(b) Calculate and plot $x_1(t)$ and $x_2(t)$ corresponding to $x_1(0) = 1$ and $x_2(0) = 0$ for $0 \le t \le 4\pi$.

9. Transform the second-order LDE

$$y'' + 2ty' + (t^2 + 2)y = 0\,, \quad y(0) = y_0\,, \quad y'(0) = y_1$$

into a system of two first order linear differential equations by defining $x_1(t) = y(t), x_2(t) = y'(t) + ty(t)$.

10. (a) Let A and C be $n \times n$ real matrices. Show that the solution of the matrix differential equation

$$X'(t) = AX(t) + X(t)C\,, \quad X(0) = X_0$$

is

$$X = e^{At} X_0 e^{Ct}$$

(b) Suppose that $C = A^t$ and $X_0 = \mathbf{v}_i \mathbf{v}_j^t$, where $\mathbf{v}_i$ and $\mathbf{v}_j$ are eigenvectors of A associated with the eigenvalues λ_i and λ_j. Find a simple expression for the solution.

11. It is given that $\mathbf{x} = \mathrm{col}\,[\,\cos t + \sin t\,,\, \cos t + 3\sin t\,]$ is a solution of a homogeneous SLDE $\mathbf{x}' = A\mathbf{x}$.

(a) Find A.

(b) Find the solution corresponding to $x_0 = \mathrm{col}\,[\,1\,,\,2\,]$.

12. Consider the SLDE in (6.18), where

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & -2 & -3 \end{bmatrix}$$

(a) Find a modal matrix and the Jordan form of A.

(b) Find the solution corresponding to $\mathbf{x}_0 = \mathrm{col}\,[\,1, 0, -1\,]$ and decompose it into its modes.

(c) Find an initial condition $\mathbf{x}_0 \neq \mathbf{0}$ such that $x_1 = \phi_1(t, \mathbf{x}_0) = 0$ for all t, where x_1 is the first component of $\mathbf{x}$.

13. Obtain the modal decomposition of the solution of the SLDE

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \mathbf{x}\,, \quad \mathbf{x}(0) = \mathbf{x}_0$$

corresponding to

(a) $\mathbf{x}_0 = \mathrm{col}\,[\,0, -1, 3\,]$

(b) $\mathbf{x}_0 = \mathrm{col}\,[\,1, -1, 2\,]$

14. Obtain the modal decomposition of the solution of the SLDE

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -7 & -3 \end{bmatrix} \mathbf{x}\,, \quad \mathbf{x}(0) = \mathbf{x}_0$$

corresponding to

(a) $\mathbf{x}_0 = \mathrm{col}\,[\,1, 1, -3\,]$

(b) $\mathbf{x}_0 = \mathrm{col}\,[\,0, 0, 2\,]$

15. Decompose the solution of (6.18) for
$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
both into complex and also into real modes.

16. Show that if the $t^q e^{\lambda_i t}$ term appears in a mode
$$\boldsymbol{\phi}(t;\mathbf{x}_{ij}) = P_{ij} e^{J_{ij} t} \boldsymbol{\alpha}_{ij}$$
of a SLDE, then all the $t^m e^{\lambda_i t}, m < q$, terms must also appear. Hint: Rewrite the expression for $\boldsymbol{\phi}(t, \mathbf{x}_{ij})$ above by partitioning P_{ij} into its columns and using the expression (6.29) for $e^{J_{ij} t}$, and show that $t^q e^{\lambda_i t}$ appears in the expression if and only if $\boldsymbol{\alpha}_{ij}$ contains a nonzero element in at least one of the positions $q+1, \ldots, n_{ij}$.

17. Let $\mathbf{V} \subset \mathbf{R}^{n\times 1}$ be an m-dimensional A-invariant subspace. Show that if $\mathbf{x}_0 \in \mathbf{V}$, then $\boldsymbol{\phi}(t;\mathbf{x}_0) \in \mathbf{V}$ for all t, where $\boldsymbol{\phi}$ denotes the solution of (6.18). Hint: Let columns of the $n \times m$ matrix V form a basis for $\mathbf{V}$. Since $\mathbf{V}$ is A-invariant, we have $AV = VF$ for some $n \times n$ matrix F. Also, if $\mathbf{x}_0 \in \mathbf{V}$ then $\mathbf{x}_0 = V\boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} \in \mathbf{R}^{m\times 1}$.

18. Rewrite each of the following LDE's as an equivalent SLDE by defining new dependent variables as in (6.33).
    (a) $y''' - 2y'' - y' + 2y = e^t\,, \quad y(0) = 1\,, \quad y'(0) = 0$
    (b) $(D^2 - 4D + 5)(D-1)(D-2)(y) = u(t)$

19. Show that the following sets of functions are linearly independent.
    (a) $\{\,1, t, \ldots, t^n\,\}$
    (b) $\{\,te^t, \cos t, \sin t\,\}$

20. (a) Show that the functions $f_1(t) = t^3$ and $f_2(t) = |\,t\,|^3$ are linearly independent on $\mathcal{I} = (-1, 1)$. Hint: Suppose that
$$c_1 f_1(t) + c_2 f_2(t) = 0$$
for all $-1 < t < 1$. Evaluate the expression at $t_1 = -0.5$ and $t_2 = 0.5$
    (b) Show that the Wronskian of f_1 and f_2 vanishes identically on $(-1, 1)$. Hint:
$$\frac{d}{dt}\,|\,t\,|^3 = 3t|\,t\,|$$

21. Show that if ϕ_1 and ϕ_2 are solutions of a second order homogeneous differential equation
$$y'' + a_1(t)y' + a_2(t)y = 0$$
then their Wronskian $w(t) = w_{\phi_1,\phi_2}(t)$ satisfies the first order homogeneous differential equation
$$w' + a_1(t)w = 0$$
Explain why this implies that either $w(t) = 0$ for all t, or $w(t) \neq 0$ for all t.

22. Solve the following initial value problems
    (a) $y'' + y' - 2y = 0\,, \quad y(0) = 3\,, \quad y'(0) = 0$
    (b) $y'' - 2y' + y = 0\,, \quad y(0) = 0\,, \quad y'(0) = 3$
    (c) $t^2 y'' - t(t+2)y' + (t+2)y = 0\,, \quad y(1) = 1\,, \quad y'(1) = 2$
    Hint: $\phi_1(t) = t$ is a solution.

(d) $(D^2 + 4D + 13)(D + 1)(y) = 0\,, \quad y(0) = 1\,, \quad y'(0) = 5\,, \quad y''(0) = 1$

23. Show that if $p > q$, then

(a) $D^p(t^q) = 0$

(b) $(D - \lambda)^p(t^q e^{\lambda t}) = 0$
Hint: First prove the result for $p = q + 1$ by using induction on q.

24. Find general solutions of the following LDE's.

(a) $y'' + y' - 2y = 2t^3$

(b) $y'' + y' - 2y = 4e^{-t}$

(c) $y'' + y' - 2y = (6t - 4)e^t$

(d) $y'' + y' - 2y = \sin t$

25. Determine the form of a particular solution of

$$(D - 1)^2(D^2 - 2D + 2)(y) = u(t)$$

for each of the following.

(a) $u(t) = t^2 - te^{2t} + \cos t$

(b) $u(t) = te^t + e^t \cos 3t$

(c) $u(t) = (t^2 - 1)e^t \sin t$

26. Show that

(a) if $\lambda = \sigma + i\omega$ is not a root of the characteristic equation $L(s) = 0$, then a particular solution of (6.47) is of the form

$$\phi_p(t) = (A_0 t^r + \cdots + A_r)e^{\sigma t} \cos \omega t + (B_0 t^r + \cdots + B_r)e^{\sigma t} \sin \omega t$$

(b) if $\lambda = \sigma + i\omega$ is a root of the characteristic equation $L(s) = 0$ with multiplicity m, then the particular solution in part (a) must be modified by multiplying it with t^m.