MAXIMUM A POSTERIORI ESTIMATION OF RADAR CROSS SECTION IN SAR IMAGES USING THE HEAVY-TAILED RAYLEIGH MODEL

Alin M. Achim, Ercan E. Kuruoglu†, and Josiane Zerubia‡*

*Department of Electrical & Electronic Engineering
University of Bristol, UK, Alin.Achim@bristol.ac.uk
†ISTI-CNR, Area della Ricerca CNR di Pisa, Italy, Ercan.Kuruoglu@isti.cnr.it
‡Projet Ariana, INRIA Sophia Antipolis, France, Josiane.Zerubia@inria.fr

ABSTRACT

We describe a novel adaptive despeckling filter for Synthetic Aperture Radar (SAR) images. In the proposed approach, the Radar Cross Section (RCS) is estimated using a *maximum a posteriori* (MAP) criterion. We first employ a logarithmic transformation to change the multiplicative speckle into additive noise. We model the RCS using the *heavy-tailed Rayleigh* distribution, which was recently proposed as an accurate model for amplitude SAR images. We estimate model parameters from noisy observations by applying the "method-of-log-cumulants", which relies on the Mellin transform. Finally, we compare our proposed algorithm with the classical Lee filtering technique applied on an aerial image and we quantify the performance improvement.

1. INTRODUCTION

SAR images are inherently affected by a signal dependent noise known as speckle, which is due to the radar wave coherence [1]. Speckle is not truly a noise in the typical engineering sense, since its texture often carries useful information about the scene being imaged. However, its presence is generally considered undesirable since it damages radiometric resolution and it affects the tasks of human interpretation and scene analysis. Thus, it appears sensible to reduce speckle in SAR images, provided that the structural features and textural information are not lost.

Many adaptive filters for SAR image denoising have been proposed in the past. The simplest approaches to speckle reduction are based on temporal averaging [1], median filtering, and homomorphic Wiener filtering [2]. The Lee MMSE filter was designed as a linear filter based on the minimum mean-square error (MMSE) criterion, optimal when both the scene and the detected intensities are Gaussian distributed and based on a linear approximation made for the multiplicative noise model [3]. Finally, the Gamma MAP filter was based on a Bayesian analysis of the image statistics where both radar cross section (RCS) and speckle noise follow a Gamma distribution [4].

In this paper we propose the use of an alternative RCS model for designing a speckle removal filter. Thus, we employ the heavy-tailed Rayleigh distribution [5] that was shown to be well justified by the physics of the radar wave scattering. Specifically, the model was developed based on the observation that the real and imaginary parts of the received complex signal can be accurately modelled using the symmetric alpha-stable family of distribution. Under the assumption of a multiplicative speckle noise model, we first employ a logarithmic transformation in order to change the noise into an additive one. Then, the general MAP solution for the resulting model is derived and the model parameter estimation is presented. The proposed estimation method is based on the second-kind statistic theory employing Mellin's transform [6] as recently proposed by Nicolas and co-workers [7].

The paper is organized as follows. In Section 2 we discuss the statistical properties of SAR images, as well as those of log-transformed images. In Section 3, we present the design of our MAP estimator based on the heavy-tailed Rayleigh signal model, which includes a novel parameter estimation method based on the Mellin transform. In Section 4, we evaluate the performance of our proposed filter and we compare it with existing speckle removal methods. Finally, in Section 5 we conclude the paper with a short summary.

2. STATISTICAL MODELING OF SAR IMAGES

Parametric Bayesian processing presupposes proper modeling for the prior probability density function (pdf) of both the radar cross section and speckle noise. In this section we briefly review the statistical properties of speckle and we describe the model used for the RCS.

2.1 Statistics of log-transformed speckle

The statistical properties of speckle noise were studied by Goodman [1]. He has shown that, if the number of scatterers per resolution cell is large, a fully developed speckle pattern can be modeled as the magnitude of a complex Gaussian field with i.i.d. real and imaginary components. A general model for speckle noise proposed by Jain [2] is constantly employed when one is concerned with the implementation of a homomorphic filter. Specifically, if we denote by $y(u,v)$ a noisy observation (i.e., the recorded SAR image envelope) of the two-dimensional function $x(u,v)$ (i.e., the noise-free SAR image that has to be recovered) and by $\quad(u,v)$the corrupting multiplicative speckle noise, one can write:

$$y(u,v) = x(u,v)\cdot \quad (u,v) \tag{1}$$

To transform the multiplicative noise model into an additive one, we apply the logarithmic function on both sides of (1):

$$Y(u,v) = X(u,v) + N(u,v), \tag{2}$$

where $Y(\cdot)$, $X(\cdot)$, and $N(\cdot)$ are the logarithms of $y(\cdot)$, $x(\cdot)$, and $\quad(\cdot)$, respectively. For a SAR image representing an average of L looks in amplitude format, the speckle noise random variable in (1) is often assumed to follow a Nakagami distribution with unit mean and variance $1/L$. This assumption is motivated by the fact that the corresponding density for the case of an image in intensity format is the widely accepted Gamma distribution [4]. The Nakagami pdf can be written as

$$p_A(\quad) = \frac{2L^L \quad ^{2L-1} e^{-L \quad ^2}}{\quad(L)} \tag{3}$$

For reasons that will become obvious within the next sections, we also provide here the first and second orders log-cumulants of a Nakagami distribution

$$\tilde{k}_{A(1)} = \frac{1}{2}(\quad(L) - \log(L))$$
$$\tilde{k}_{A(2)} = \frac{1}{4}\quad(1,L) \tag{4}$$

where is the Digamma function and $\quad(r,L)$ is the Polygamma function, i.e. the r-th derivative of the Digamma function. Having in mind that $p(\ )d\ \ = p(N)dN$, one can readily obtain the pdf of the random variable $N = \log$

$$p_A(N) = \frac{2L^L e^{2NL} e^{-Le^{2N}}}{(L)} \qquad (5)$$

2.2 The generalized Rayleigh model

The SAR image formation theory has been long time dominated by the assumption of Gaussianity for the real and imaginary parts of the received complex signals. Based on this assumption, the detected amplitude SAR images can be modeled by a Rayleigh distribution. However, as we will show in this section, invoking a generalized version of the central limit theorem, the assumption of Gaussianity can be replaced by an assumption of "alpha-stability" resulting in a more powerful model for the detected amplitude pdf. In the following we provide a brief, necessary overview of the alpha-stable statistical model on which the generalized Rayleigh pdf is actually based.

2.2.1 Symmetric Alpha-Stable Distributions

The $S\ \ S$ distribution lacks a compact analytical expression for its probability density function (pdf). Consequently, it is most conveniently represented by its characteristic function [8]

$$(\ ) = \exp(j\ \ \ \ -\ |\ \ |\ \ ) \qquad (6)$$

where is the *characteristic exponent*, taking values $0 < \ \ \leq 2$, $(- \ \ < \ \ < \ )$ is the *location parameter*, and $(\ \ > 0)$ is the *dispersion* of the distribution. For values of in the interval $(1,2]$, the location parameter corresponds to the mean of the $S\ \ S$ distribution, while for $0 < \ \ \leq 1$, corresponds to its median. The dispersion parameter determines the spread of the distribution around its location parameter , similar to the variance of the Gaussian distribution.

The characteristic exponent is the most important parameter of the $S\ \ S$ distribution and it determines the shape of the distribution. The smaller the characteristic exponent is, the heavier the tails of the $S\ \ S$ density. This implies that random variables following $S\ \ S$ distributions with small characteristic exponents are highly impulsive. One consequence of heavy tails is that only moments of order less than exist for the non-Gaussian alpha-stable family members. As a result, stable laws have infinite variance. Gaussian processes are stable processes with $= 2$ while Cauchy processes result when $= 1$.

2.2.2 A Heavy-Tailed Rayleigh model

Kuruoglu and Zerubia [5] assumed that the real and imaginary parts of the received SAR signal are jointly $S\ \ S$. Consequently, they derived the following integral equation for the amplitude pdf of SAR images, which they called the *heavy-tailed Rayleigh distribution*:

$$p(x) = x \int_0 \ \ u \exp(-\ \ u\ \ ) J_0(ux) du \qquad (7)$$

where J_0 is the zeroth order Bessel function of the first kind.

It is important to note at this point, that by considering the special case $= 2$, we obtain

$$p(x) = \frac{x}{2} \exp(-\frac{x^2}{4}) \qquad (8)$$

which is basically the classical Rayleigh distribution as expected since for $= 2$ the $S\ \ S$ distribution reduces to Gaussian. Also, by taking $= 1$ in (7), one obtain the following pdf, which we will refer to as the Cauchy-Rayleigh model

$$p(x) = \frac{x}{(x^2 + \ ^2)^{3/2}} \qquad (9)$$

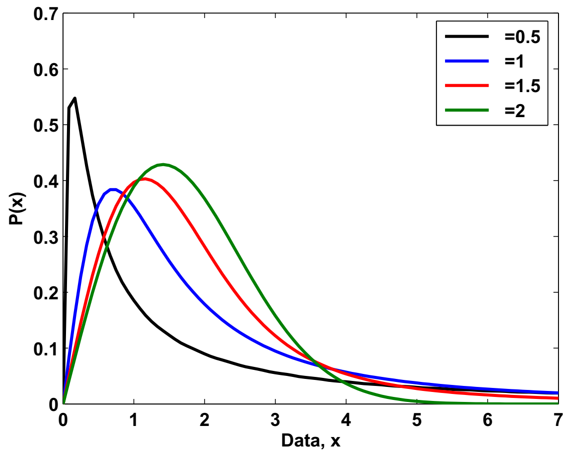

Figure 1: Generalized Rayleigh probability density functions for different values of the characteristic exponent . The dispersion parameter is kept constant at $= 1$.

In Fig. 1 we show the tail behavior of several heavy-tailed Rayleigh densities including the particular cases corresponding to the Cauchy and the Gaussian distributions. In our further developments we will employ a homomorphic transformation in order to transform the multiplicative speckle noise into an additive one. Consequently, let us also provide here the logarithmic domain pdf corresponding to the heavy-tailed Rayleigh

$$p_A(X) = e^{2X} \int_0 \ \ u \exp(-\ \ u\ \ ) J_0(ue^X) du \qquad (10)$$

where $X = \ln x$.

3. ADAPTIVE FILTERING OF SPECKLE NOISE

After applying a logarithmic transformation to the original data we get an image represented as the sum of the transformations of the signal and of the noise:

$$Y = X + N \qquad (11)$$

The MAP estimator of X given the noisy observation Y is:

$$\hat{X}(Y) = \arg\max_X P_{X|Y}(X|Y) \qquad (12)$$

Bayes' theorem gives the *a posteriori* PDF of X based on the measured data:

$$P_{X|Y}(X|Y) = \frac{P_{Y|X}(Y|X) P_X(X)}{P_Y(Y)}, \qquad (13)$$

where $P_X(X)$ is the *prior* PDF of the signal component of the measurements and $P_{Y|X}(Y|X)$ is the *likelihood* function. Substituting (13) in (12), we get:

$$\hat{X}(Y) = \arg\max_X P_{Y|X}(Y|X) P_X(X) = \arg\max_X P_N(Y-X) P_X(X) = \arg\max_X P_N(N) P_X(X) \qquad (14)$$

In the above equation we use a heavy-tailed Rayleigh model for the signal component, while we use a Nakagami model for the noise component. Naturally, in order for the processor in Eq. (14) to be of any practical use, one should be able to estimate the parameters $_X$ and $_X$ of the signal from the observed data. In the next subsection we derive parameter estimation methods for the generalized Rayleigh pdf based on second-kind cumulants.

3.1 Parameter estimation using Mellin transform

Following the arguments in [6], Nicolas has recently proposed the use of Mellin transform as a powerful tool for deriving parameter estimation methods based on log-cumulants for the case of multiplicative noise contamination as is the case with SAR images [7]. In the following, we briefly review the Mellin transform and its main properties that we used in our derivations.

3.1.1 Mellin transform

Let f be a function defined over $^{+}$. The integral transform defined by

$$(s) = \mathbf{M}[f(u)](s) = \int_0^{+} u^{s-1} f(u) du \qquad (15)$$

is called the Mellin transform of f. The inverse transform is given by

$$f(u) = \mathbf{M}^{-1}[\ (s)](u) = \frac{1}{2\ j} \int_{c-j}^{c+j} u^{-s}\ (s) ds \qquad (16)$$

The transform (s) exists if the integral $\int_0^{+} |f(x)| x^{k-1} dx$ is bounded for some $k > 0$, in which case the inverse $f(u)$ exists with $c > k$. The functions (s) and $f(u)$ are called a Mellin transform pair, and either can be computed if the other is known.

By analogy with the way in which common statistics are deducted based on Fourier Transform, the following second-kind statistic functions can be defined, based on Mellin Transform [7]

- Second-kind first characteristic function

$$(s) = \int_0^{+} x^{s-1} p(x) dx \qquad (17)$$

- Second-kind second characteristic function

$$(s) = \log(\ (s)) \qquad (18)$$

- r^{th} order second-kind moments

$$\tilde{m}_r = \frac{d^r\ (s)}{ds^r}\Big|_{s=1} = \int_0^{+} (\log x)^r p(x) dx \qquad (19)$$

- r^{th} order second-kind cumulants

$$\tilde{k}_r = \frac{d^r\ (s)}{ds^r}\Big|_{s=1} \qquad (20)$$

The first two second-kind cumulants can be estimated empirically from N samples y_i as follows

$$\hat{\tilde{k}}_1 = \frac{1}{N} \sum_{i=1}^{N} [\log(y_i)]$$

$$\hat{\tilde{k}}_2 = \frac{1}{N} \sum_{i=1}^{N} [(\log(y_i) - \hat{\tilde{k}}_1)^2] \qquad (21)$$

- Finally, for two functions f and g, Mellin's convolution is defined over the interval $[0,\ ]$ as

$$(f \hat{*} g)(y) = \int_0^{+} f(x) g(\frac{y}{x}) \frac{dx}{x} = \int_0^{+} f(\frac{y}{x}) g(x) \frac{dx}{x} \qquad (22)$$

3.1.2 Log-moment estimation of the generalized Rayleigh model

By plugging the expression of the heavy-tailed Rayleigh pdf given by (7) into (17) and after some straightforward manipulations, details of which can be found in [5], one gets

$$(s) = \frac{2^s\ (\frac{s+1}{2})\ ^{\frac{s-1}{}}\ (\frac{1-s}{})}{(\frac{1-s}{2})} \qquad (23)$$

which is the second-kind first characteristic function of the heavy-tailed Rayleigh density. Kuruoglu and Zerubia [5] used this expression for two different values of s and subsequently solved the resulting system in order to get estimates of the parameters and . However, here we are interested in deriving estimates of the model parameters in the case of multiplicative noise contamination. Consequently, we settled by plugging the above expression in (18) and subsequently in (20), thus obtaining the following results for the second-kind cumulants of the model

$$\tilde{k}_{A(1)} = -\ (1)\frac{1-}{} + \log(2^{\ \frac{1}{}})$$

$$\tilde{k}_{A(2)} = \frac{(1,1)}{2} \qquad (24)$$

Using the above system of two equations one can readily solve for the parameters and of the heavy-tailed Rayleigh distribution. Remember however that our measurement is a mixture of heavy-tailed Rayleigh signal and Nakagami distributed speckle noise. Under the multiplicative speckle noise model (1), if we denote by $p_y(y)$, $p_x(x)$, and $p\ (\ )$ the pdfs of y, x, respectively, it can be shown that the pdf of y is given in fact by the Mellin convolution between the pdfs of x and . Consequently, the second-kind cumulant of any order of y is given by the sum of the second-kind cumulants of the same order of x and

$$\tilde{k}_{y(r)} = \tilde{k}_{x(r)} + \tilde{k}_{\ (r)} \qquad (25)$$

Using expressions (4) and (24) in the above equation together with the empirical log-cumulants in (21) we obtain the following estimates for the parameters of the heavy-tailed Rayleigh model (7) mixed with Nakagami distributed speckle noise

$$\hat{} = \sqrt{\frac{(1,1)}{\hat{\tilde{k}}_{(2)} - \frac{1}{4} \cdot\ (1,L)}}$$

$$\hat{} = [\frac{\exp(\hat{\tilde{k}}_{(1)} +\ (1)\frac{1-}{} - \frac{1}{2}(\ (L) - \log(L))}{2}] \qquad (26)$$

4. EXPERIMENTAL RESULTS

In this section, we present simulation results obtained by processing a test image using our proposed MAP speckle filter based on the heavy-tailed Rayleigh prior. Specifically, we degraded an original "speckleless" image with synthetic speckle in amplitude format. For this purpose, an aerial image was chosen due to its identical content with real SAR images. This image was obtained by cropping the *"westaerialconcorde"* image that can be found in Matlab's Image Processing Toolbox. In our experiments, we considered two different levels of simulated speckle noise corresponding to L =3 and 12 (cf. eq. 3). We compared the results of our approach with those obtained using other classical speckle filters including the median, the homomorphic Wiener, and the Lee filter.

In order to assess the quality of our proposed filter we computed two different measures based on the original and the denoised data. A common way to evaluate the noise suppression in case of multiplicative contamination is to calculate the signal-to-mean squared error (S/MSE) ratio, defined as

$$S/MSE = 10 \log_{10}(\sum_{i=1}^{K} S_i^2 / \sum_{i=1}^{K} (\hat{S}_i - S_i)^2) \qquad (27)$$

where S is the original image, $\hat{S}$ is the denoised image, and K is the image size. This measure corresponds to the classical SNR in the case of additive noise.

In addition to the above quantitative performance measure, we also consider a qualitative measure for edge preservation (e.g. [9])

$$= \frac{(\ S - \overline{\ S},\ \widehat{\ S} - \overline{\widehat{\ S}})}{\sqrt{(\ S - \overline{\ S},\ S - \overline{\ S}) \cdot (\ \widehat{S} - \overline{\widehat{\ S}},\ \widehat{S} - \overline{\widehat{\ S}})}} \qquad (28)$$

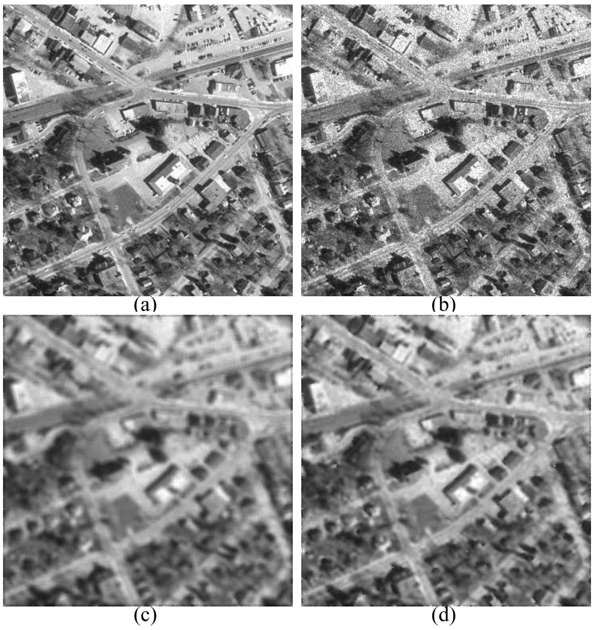

Figure 2: *Results of various speckle suppressing methods. (a) Original image. (b) Simulated speckle image ($L = 3$, amplitude format). (c) Lee filter. (d) Proposed MAP filter.*

where S and $\widehat{S}$ are the high-pass filtered versions of S and $\hat{S}$ respectively, obtained with a 3×3-pixel standard approximation of the Laplacian operator, and

$$(S_1, S_2) = \overset{K}{\underset{i=1}{}} S_{1_i} \cdot S_{2_i}. \qquad (29)$$

The correlation measure, should be close to unity for an optimal effect of edge preservation. The obtained values of S/MSE, and for all methods applied to our test image are given in Table 1. From the table it can be seen that, in most situations, our proposed filter exhibits the best performance according to both metrics. Figure 2 shows a representative result from the processing of the aerial test image. The image in Figure 2(b) was obtained by degrading the original test image (2(a)) with Nakagami distributed speckle noise (cf. eq. 3) with $L = 3$ looks. From the figure it can be seen that all the tested filters achieved a good speckle surpressing performance. However, clearly our homomorphic MAP filter based on the heavy-tailed Rayleigh signal prior did the best job in preserving the structural features that can be observed in the original image.

Table 1: Image enhancement measures obtained by four denoising methods applied on the "aerial" image. Two levels of noise are considered corresponding to ENL=3 and 12. The S/MSE of each despeckled image is given in dB.

	ENL = 3		ENL = 12	
Method	S/MSE		S/MSE	
Median	15.52	0.3309	15.88	0.3677
Wiener	15.27	0.3812	15.60	0.4150
Lee	15.35	0.3540	17.28	0.6186
proposed	16.83	0.3541	18.68	0.6471

5. SUMMARY

We presented a new homomorphic statistical filter for speckle noise removal in SAR images, which is based on the recently introduced heavy-tailed Rayleigh model for the amplitude of the RCS. Under the assumption of a multiplicative speckle noise model, we first employed a logarithmic transformation in order to change the noise into an additive one and to differentiate its characteristics from the signal characteristics. Then, a maximum a posteriori processor was implemented numerically and the corresponding nonlinearities were applied to the observed data. A novel parameter estimation method was developed for the case of generalized Rayleigh signal mixed with Nakagami distributed speckle noise. The estimation method is based on the recently proposed method-of-log-cumulants employing Mellin's transform. Our simulations results showed that the homomorphic MAP filter based on the heavy-tailed Rayleigh model is among the best for speckle removal.

6. ACKNOWLEDGEMENT

The first author would like to thank ERCIM (European Research Consortium in Informatics and Mathematics) for partial financial support during his post-doc at ISTI-CNR, Pisa, Italy and INRIA, Sophia Antipolis, France.

REFERENCES

[1] J. W. Goodman, "Some fundamental properties of speckle," *J. Opt. Soc. Amer.*, vol. 66, pp. 1145–1150, November 1976.

[2] A. K. Jain, *Fundamental of Digital Image Processing*. NJ: Prentice-Hall, 1989.

[3] J. S. Lee, "Digital image enhancement and noise filtering by use of local statistics," *IEEE Trans. on Pattern Anal. and Machine Intell.*, vol. 2, pp. 165–168, 1980.

[4] A. Baraldi and F. Parmigiani, "A refined Gamma MAP SAR speckle filter with improved geometrical adaptivity," *IEEE Trans. on Geosci. and Remote Sensing*, vol. 33, pp. 1245–1257, Sept. 1995.

[5] E. E. Kuruoglu and J. Zerubia, "Modeling SAR images with a generalization of the Rayleigh distribution," *IEEE Trans. on Image Processing*, vol. 13, pp. 527–533, April 2004.

[6] B. Epstein, "Some applications of the Mellin transform in statistics," *The Annals of Mathematical Statistics*, vol. 19, pp. 370–379, Sep 1948.

[7] C. Tison, J. M. Nicolas, F. Tupin, and H. Maitre, "A new statistical model for Markovian classification of urban areas in high resolution SAR images," *IEEE Trans. on Geosci. and Remote Sensing*, vol. 42, pp. 2046–2057, Oct. 2004.

[8] C. L. Nikias and M. Shao, *Signal Processing with Alpha-Stable Distributions and Applications*. New York: John Wiley and Sons, 1995.

[9] A. Achim, P. Tsakalides, and A. Bezerianos, "SAR image denoising via Bayesian wavelet shrinkage based on heavy-tailed modeling," *IEEE Trans. Geosc. and Remote Sensing*, vol. 41, pp. 1773–1784, Aug. 2003.